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HISTORY

*Loria, Gino. **Guida allo Studio della Storia delle Matematiche. Generalità. Didattica. Bibliografia. Appendice: Questioni Storiche Concernenti le Scienze Esatte.** 2d ed. Ulrico Hoepli, Milan, 1946. xix+385 pp. 550 lire.

Although it is not difficult to find gaps in specific sections, the overall completeness of this bibliographical work is remarkable. The grouping of the material is clearly dictated by practical experience and makes this "guide" a very useful tool for bibliographical purposes. *O. Neugebauer.*

*Bell, E. T. **The Development of Mathematics.** 2d ed. McGraw-Hill Book Company, Inc., New York, 1945. xiii+637 pp. $5.00.

This edition differs from the first, which appeared in 1940 [cf. these Rev. 2, 113], in that about fifty pages of new material have been added. The additions include numerous short amplifications of miscellaneous topics together with longer notes on subjects in which there have been striking recent advances, such as symbolism, algebraic and differential geometry and lattices. We can therefore refer to our review of the first edition and add gratefully that Laplace's "Theory of Probability" has received a graceful evaluation in the new edition. *D. J. Struik* (Cambridge, Mass.).

*Coolidge, Julian Lowell. **A History of the Conic Sections and Quadric Surfaces.** Oxford University Press, 1945. xi+214 pp. $6.00.

This work contains an introduction to the theory of conics and quadric surfaces following the main lines of the historical development. The Greek contributions to the subject are treated in a concise way, partly in the semi-modern notation used by Heath, partly in modern algebraic symbolism. The renewed interest in conics in the seventeenth century gives rise to two different methods of approach, the synthetic method of Desargues and the analytic one of Descartes, which continue to compete down to our time. The first one produced in due time the great projective school, with which the names of Poncelet, Chasles, Steiner and von Staudt are connected; the second one, which seemed to have come to a standstill in the works of L'Hospital and Euler, was enabled to maintain the competition by the introduction of new algebraic techniques (Lamé, Plücker) and the extension of the concept of coordinates (Möbius, Plücker). It was supported in an effective way by the creation of the theory of invariants (Boole, Salmon). The description of these two movements is followed by some chapters on disconnected topics: maxima and minima, closure problems, curvature, parametric representation, areas and lengths, mechanical construction of conics. Next comes an exposition of systems of conics and of conics in space. The remaining chapters are devoted to the theory of quadric surfaces; again the synthetic method (divided into an informal and a formal period) is to be distinguished from the algebraic line of approach which leads to higher algebraic domains in the treatment of pairs of quadrics and linear systems. The short concluding chapter contains some discussion of the differential properties of quadric surfaces. An index of authors quoted gives the full titles of the works referred to in the text. *E. J. Dijksterhuis* (Oisterwijk).

*Neugebauer, O., and Sachs, A. J. **Mathematical Cuneiform Texts. With a chapter by A. Goetze.** American Oriental Series, Volume 29. American Oriental Society and the American Schools of Oriental Research, New Haven, Conn., 1945. x+177 pp.+49 plates. $5.00.

Edition of all mathematical cuneiform texts discovered (chiefly in American collections) since the publication of Neugebauer's Mathematische Keilschrifttexte [Springer, Berlin, 1935]. For all texts a transcription, translation and commentary are given; a number of photographs and copies are added. Among the table texts are two new types: a table of approximate reciprocals of "irregular" numbers, correct to three or four sexagesimals, and a table of exponents of powers of two. Among the problem-texts the most remarkable is one containing a list of seventeen right-angled "Pythagorean triangles" with whole numbers l, b, d as sides, ordered according to decreasing values of b/l. The first column gives the square b^2/l^2 or possibly, as the authors suppose, d^2/l^2, the next two columns b and d. All numbers are calculated according to the classical formulas $l=2pq$, $b=p^2-q^2$, $d=p^2+q^2$, where p and q are simple numbers consisting of factors 2, 3, 5 only. Another text gives a very good approximation to $\sqrt{2}$, namely, 1; 24, 51, 10. Other texts, dealing with bricks, digging of canals, etc., contain valuable information about metrological questions, wages, dimensions of bricks, etc. Two texts contain lists of coefficients used in diverse applications. Goetze has contributed a chapter on the dialects of the Old Babylonian texts.

B. L. van der Waerden (Laren).

Sachs, A. Notes on fractional expressions in Old Babylonian mathematical texts. J. Near Eastern Studies 5, 203–214 (1946).

The great advantage of Babylonian arithmetic over Egyptian or Greek methods of computation lies in the consistent use of sexagesimal expansions for fractions in contrast to decompositions into sums of unit fractions. However, the newly discovered text published here proves the existence of similar more primitive methods for Babylonia also. Metrological and historical implications are discussed.

O. Neugebauer (Providence, R. I.).

Cassina, U. Sulla geometria egiziana. Period. Mat. (4) 22, 1–29 (1942).

Discussion of the geometrical examples found in the Rhind Papyrus and the Moscow Mathematical Papyrus. The author has apparently overlooked a similar paper of the reviewer, published in Quellen und Studien zur Geschichte der Mathematik, Abt. B. 1, 413–451 (1931).

O. Neugebauer (Providence, R. I.).

Marar, K. Mukunda, and Rajagopal, C. T. Gregory's series in the mathematical literature of Kerala. Math. Student 13, 92–98 (1945).

Reprint of part of a paper which appeared in J. Bombay Branch, Roy. Asiatic Soc. (N.S.) 20, 65–82 (1944); these Rev. 6, 253.

Bèttica-Giovannini, Renato. Problemi di geometria in Platone. Period. Mat. (4) 22, 129–144 (1942).

Discussion and translation of mathematical passages in the dialogue Menon. *O. Neugebauer* (Providence, R. I.).

Beumer, M. G. Archimedes and the trisection of the angle. Nieuw Tijdschr. Wiskunde 33, 281–287 (1946). (Dutch)

Beth, Evert W. Gorgias of Leontini as philosopher. Algemeen Nederlandsch Tijdschrift voor Wijsbegeerte en Psychologie 36, 41–58 (Annalen van het Genootschap voor Wetenschappelijke Philosophie 12, 1–18) (1943). (Dutch)

Scharfe Analyse der erhaltenen Gorgias-Exzerpte.

H. Freudenthal (Amsterdam).

Beth, E. W. Notes on the history of philosophy. Algemeen Nederlandsch Tijdschrift voor Wijsbegeerte en Psychologie 36, 80–83 (1942). (Dutch)

Im zweiten Teil dieses Artikels behandelt der Verfasser Spuren der Kalkül-Auffassung der Logik vor Leibniz, besonders bei Aristoteles. *H. Freudenthal* (Amsterdam).

***Zeller, Mary Claudia. The Development of Trigonometry from Regiomontanus to Pitiscus.** Dissertation, University of Michigan, 1944. vi+119 pp.

Chapter I deals with trigonometrical works before Regiomontanus and contains nothing new. The author mentions some Greek mathematicians and in the traditional way begins with Thales "who spent many years in Egypt. . . ." She allows only one page to Menelaus and Ptolemy. The new edition of Menelaus [M. Krause, Abh. Ges. Wiss. Göttingen, Phil.-Hist. Kl. (3) no. 17 (1936)] is not mentioned. The thesis proper begins with chapter II. The reader is introduced to a great many facts and details, but no attempt has been made to present this material according to some leading idea. It would, for example, be of interest to learn something about the structure of the different trigonometrical tables, how they were computed and in what respect they differ from Ptolemy's tables of chords. The book will be of value as a bibliography on works of trigonometry from the fifteenth and sixteenth centuries.

O. Schmidt (Copenhagen).

de Vries, Hk. Historical studies. XXIV. On the contact and intersection of circles and conic sections. Nieuw Tijdschr. Wiskunde 33, 100–164 (1946). (Dutch) [MF 16910]

Duarte, F. J. On the non-Euclidean geometries. Historical and bibliographical notes. Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. 9, 1–67 (5 plates) (1945). (Spanish)

Karpinski, Louis C. The place of trigonometry in the development of mathematical ideas. Scripta Math. 11, 268–272 (1945).

Conte, Luigi. Esiste un "metodo di Cartesio" per la risoluzione dell'equazione biquadratica? Period. Mat. (4) 23, 1–11 (1943).

The question was raised by G. Candido [Period. Mat. (4) 21, 21–44 (1941), in particular, p. 33; these Rev. 3, 97] and answered in the negative. According to Candido the real father of the method which goes by the name of Descartes is Ferrari, while the first demonstration was given by van Schooten. Conte vindicates the rights of Descartes and repels the more or less veiled accusation of plagiarism which is involved in the paper of Candido. *E. J. Dijksterhuis.*

Kaval'eri [Cavalieri], Bonaventura. Geometriya Izlozhennaya Novym Sposobom pri Pomoščі Nedelimyh Nepreryvnogo, s Priloženiem "Opyta IV" o Primenenii Nedelimyh k Algebraičeskim Stepenyam. Tom Pervyĭ. Osnovy Učeniya o Nedelimyh ("Geometriya," Kn. I i II i "Opyt IV"). [Geometry Expounded by a New Method with the Use of Indivisible Continua, with the Addition of "Essay IV" on the Application of Indivisibles to Algebraic Powers. Vol. I. Foundations of the Theory of Indivisibles ("Geometry," Books I and II and "Essay IV")]. Translation, with introduction, notes and commentary by S. Ya. Lur'e. Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, Moscow-Leningrad, 1940. 416 pp. (2 plates) (Russian)

The translation is of books I and II of Geometria Indivisibilibus Continuorum Nova Quadam Ratione Promota, Bologna, 1635, and of book IV of Exercitationes Geometricae Sex, Bologna, 1647. The introduction, pages 9–80, deals with the mathematical work of Cavalieri in general.

***Lobačevskiĭ, N. I. Geometričeskie Issledovaniya po Teorii Parallel'nyh Liniĭ. [Geometric Investigations on the Theory of Parallel Lines].** Translation, commentary, introduction and notes by V. F. Kagan. Akademiya Nauk SSSR, Moscow-Leningrad, 1945. 176 pp. (Russian)

Having failed to interest his Russian contemporaries in his geometry, Lobačevskiĭ made an attempt to acquaint the mathematicians of western Europe with it; in 1840 he published a booklet in German containing a short exposition of his theory (it is known that Gauss read it and recognized the merit of Lobačevskiĭ's work). The present publication contains a Russian translation of the booklet which occupies 41 pages; it is preceded by 30 pages of introductory material giving the historical background and is followed by about 90 pages containing extensive notes which explain and amplify many passages of the text, and two appendices. One appendix gives some additional material in non-Euclidean geometry rounding out, so to speak, Lobačevskiĭ's very concise exposition; the other gives some more modern proofs made possible by the development of mathematics during the last hundred years. *G. Y. Rainich.*

***Euler, Leonhard. Commentationes Arithmeticae. Volumen Quartum edidit Rudolf Fueter. (Opera Omnia. Series Prima: Opera Mathematica, Volumen Quintum.)** Societas Scientiarum Naturalium Helveticae, Geneva, 1944. xlvii+374 pp.

The first half of this volume contains papers on Diophantine equations which were published posthumously by the St. Petersburg Academy. The second half contains extracts from Euler's notebooks and notes by his students; some of this material was published in Commentationes Arith-

meticae Collectae [1849]. The introduction contains an account of the contents [29 pp.] by Fueter and "Übersicht über die Zahlentheorie in Eulers Algebra" [7 pp.] by Speiser.

Euler, Leonhard. Introductio in Analysin Infinitorum. Tomus Secundus edidit Andreas Speiser. (Opera Omnia. Series Prima: Opera Mathematica, Volumen Nonum.) Societas Scientiarum Naturalium Helveticae, Geneva, 1945. 1+403 pp.

This volume contains, in addition to its own introduction, an introduction to vol. 10 of the same series [Institutiones Calculi Differentialis].

Archibald, Raymond Clare. Mathematical table makers—portraits, paintings, busts, monuments, bio-bibliographical notes. I. Scripta Math. **11**, 213–245 (11 plates) (1945).

Archibald, Raymond Clare. Mathematical table makers—portraits, paintings, busts, monuments, bio-bibliographical notes. II. Scripta Math. **12**, 15–51 (4 plates) (1946).

***Lur'e, S. Ya. Arhimed. [Archimedes].** Scientific-Popular Series of Biographies. Akad. Nauk SSSR, Moscow-Leningrad, 1945. 272 pp. (Russian)

Bell, E. T. Obituary: Harry Bateman. Quart. Appl. Math. **4**, 105–111 (1946). [MF 16958]

Morse, Marston. George David Birkhoff and his mathematical work. Bull. Amer. Math. Soc. **52**, 357–391 (1946). [MF 16902]

Langer, R. E. Obituary: George David Birkhoff. Trans. Amer. Math. Soc. **60**, 1–2 (1946).

Whittaker, E. T. Obituary: George David Birkhoff. J. London Math. Soc. **20**, 121–128 (1945). [MF 16704]

Carathéodory, Constantin. Obituary: David Hilbert. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1943**, 350–354 (1944).

de Broglie, Louis. La vie et l'oeuvre de M. Émile Picard. Mém. Acad. Sci. Inst. France (2) **66**, 45 pp. (1943).

Burkill, J. C. Obituary: Samuel Pollard. J. London Math. Soc. **20**, 189–192 (1945).

Scientific works of Doctor Julio Rey Pastor in the period 1905–1945. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 355–377 (1946). (Spanish) [MF 16937]

Severi, Francesco. L'opera scientifica di Gaetano Scorza. Ann. Mat. Pura Appl. (4) **20**, 1–20 (1941). [MF 16595]

Reeve, William David. Obituary: David Eugene Smith. Scripta Math. **11**, 209–212 (2 plates) (1945).

Pérard, Albert. Obituary: Vito Volterra (1860–1940). Cahiers de Physique no. **3**, 51–58 (1941).

FOUNDATIONS

Sierpinski, Waclaw. Sur une proposition équivalente à l'axiome du choix. Actas Acad. Ci. Lima **9**, 111–112 (1946).

On démontre que l'axiome du choix est équivalent à la proposition suivante: tout nombre cardinal qui n'est pas un nombre naturel n'est pas le produit de deux nombres cardinaux plus petits que lui. *T. Viola* (Rome).

Mostowski, Andrzej. Axiom of choice for finite sets. Fund. Math. **33**, 137–168 (1945). [MF 16627]

This paper is concerned with finding necessary and sufficient conditions under which, given a positive integer n and a finite set of positive integers $Z=\{n_1, \cdots, n_k\}$, the proposition that $[Z]\rightarrow[n]$ holds, where $[n]$ is the proposition that for every class K of sets with n elements there is a function f_K (the choice function for K) defined for all $X \varepsilon K$ and such that $f_K(X) \varepsilon X$, and $[Z]$ is the logical product of the k propositions $[n_1], \cdots, [n_k]$. It is easy to prove that $[2]\rightarrow[4]$ and to disprove that $[3]\rightarrow[4]$.

The methods of this paper are primarily group-theoretical, and somewhat similar to methods employed by Fraenkel [J. Symbolic Logic **2**, 1–25 (1937)]. Let S_n be the symmetric group of degree n. A subgroup G of S_n will be said to have no fixed points if, for any $i \leqq n$, there is an $f \varepsilon G$ such that $f(i) \neq i$. The index of the subgroup H of any group G will be Ind (G/H). The author proves that $[Z]\rightarrow[n]$ holds under the (sufficient) condition that every subgroup G of S_n without fixed points contains a subgroup H such that there is a finite number r of (not necessarily distinct) proper subgroups $K_1, \cdots, K_r$ of H, where the sum Ind (H/K_1) $+\cdots+$Ind (H/K_r) is a member of Z. He also proves that the condition that, for every decomposition $n=p_1+\cdots+p_s$ of n into a sum of (not necessarily distinct) primes, there is in X a number divisible by at least one of the primes p_i, is also sufficient for $[Z]\rightarrow[n]$.

It is also shown that the following conditions are necessary for $[Z]\rightarrow[n]$. (1) For every subgroup G of S_n without fixed points there is a group $H \subset G^{\mathfrak{N}_0}$ and a finite number r of (not necessarily distinct) proper subgroups $K_1, \cdots, K_r$ of H such that the sum Ind $(H/K_1)+\cdots+$Ind (H/K_r) is a member of Z, where $G^{\mathfrak{N}_0}$ is a certain subgroup of $G^{\mathfrak{N}_0}$, the set of all infinite sequences $\varphi_1, \varphi_2, \cdots$, where the φ_i's are members of G. The proof consists in part in reinterpreting the axioms of set theory in such a way that these axioms together with $[Z]$ become true, whereas $[n]$ becomes false. (2) For any decomposition of n into a sum of primes $p_1+\cdots+p_s$ there are s nonnegative integers $q_1, \cdots, q_s$ such that the sum $p_1q_1+\cdots+p_sq_s$ is a member of Z. Further particular cases under which $[Z]\rightarrow[n]$ are also considered. *R. M. Martin* (Bryn Mawr, Pa.).

Mautner, F. I. An extension of Klein's Erlanger program: logic as invariant-theory. Amer. J. Math. **68**, 345–384 (1946).

The thesis is developed that the Boolean propositional calculus can be considered as the invariant theory of the symmetric group, using truth functions for coordinates. The author shows that H. Weyl's axioms for Klein's Erlanger program [The Classical Groups, Their Invariants and Representations, Princeton University Press, 1939, pp. 17–18; these Rev. 1, 42] are fulfilled. In showing this he develops a Boolean tensor algebra, with invariants, covari-

ants (including quantifiers) and matrices. He shows that all Boolean invariants, under the symmetric group on Boolean tensors, can be obtained by the calculus of propositional functions. *G. Birkhoff* (Cambridge, Mass.).

Skolem, Th. Remarks on recursive functions and relations. Norske Vid. Selsk. Forh. **17**, no. 22, 89–92 (1944). [MF 16888]

In this and the three papers reviewed below Skolem contributes some details to the theory of recursive arithmetical functions. S. C. Kleene [Math. Ann. **112**, 727–742 (1936), in particular, p. 736] showed that every general recursive [this term is defined by Kleene, loc. cit.] function $f(x_1, \cdots, x_n)$ is expressible in the form

$$\psi[\epsilon y R(x_1, \cdots, x_n, y)], \tag{1}$$

where ψ is a primitive recursive function, R is a primitive recursive relation which is satisfied, for given $x_1, \cdots, x_n$, by at least one y, and $\epsilon y R(x_1, \cdots, x_n, y)$ is the least such y. Skolem raises the question as to what sorts of recursive functions are so expressible without a ψ, that is, in the form (2) $\epsilon y S(x_1, \cdots, x_n, y)$, where S is a primitive recursive relation satisfying the same hypotheses as R. In the present paper he proves three theorems along this line. (I) If the ψ in (1) is steadily increasing, the form (1) can be reduced to the form (2). (II) It is always possible to find two primitive recursive relations S_1 and S_2 so that

$$\psi(\epsilon y R(x_1, \cdots, x_n, y)) = \epsilon z S_1(x_1, \cdots, x_n, y) - \epsilon z S_2(x_1, \cdots, x_n, y).$$

(III) If $\phi(x_1, x_2)$ is the doubly recursive function defined by R. Péter [Math. Ann. **111**, 42–60 (1935), in particular, p. 59; the function is a modification of one introduced by Ackermann, Math. Ann. **99**, 118–133 (1928)] then the relation $y = \phi(x_1, x_2)$ is primitive recursive, and so is itself an S suitable for (2) with the additional property that y is unique. *H. B. Curry* (State College, Pa.).

Skolem, Th. Some remarks on recursive arithmetic. Norske Vid. Selsk. Forh. **17**, no. 26, 103–106 (1944). [MF 16889]

Continuing the theme of the paper reviewed above, Skolem proves the following three theorems. (I) The function $f(x_1, \cdots, x_n)$ is expressible in the form (2) [in the preceding review] when and only when the relation

$$y = f(x_1, \cdots, x_n) \tag{3}$$

is primitive recursive. (II) There exist numerical functions with values restricted to 0 and 1 which are recursive but not primitive recursive. (III) There is no general recursive method for deciding, for all primitive recursive relations $R(x \cdot y)$, whether $(x)(Ey)R(x \cdot y)$ is true or not. The third theorem was proved by Kleene [reference in the preceding review] but the author derives the result quite simply from Kleene's theorem on expressibility in the form (1). In regard to the second theorem, the author remarks that the question remains open as to whether there exists a relation of form (3) (with f general recursive) which is not primitive recursive. *H. B. Curry* (State College, Pa.).

Skolem, Th. Some remarks on the comparison between recursive functions. Norske Vid. Selsk. Forh. **17**, no. 32, 126–129 (1945). [MF 16890]

The theme of this note is somewhat different from that of the two reviewed above. Skolem proved [Fund. Math. **23**, 150–161 (1934); see also Norsk. Mat. Forenings Skr. (2), no. **10**, 73–82 (1933)] that, if $\{f_n(x)\}$ is a sequence of arithmetical functions (that is, functions of natural numbers taking values which are also natural numbers), then there exist arithmetical functions $g(x)$ and $h(m, n)$, $g(x)$ steadily increasing, such that for arbitrary m, n one of the relations $f_m(x) > f_n(x)$, $f_m(x) = f_n(x)$, $f_m(x) < f_n(x)$ holds for all $x = g(z)$ when $z \geqq h(m, n)$. Skolem considers here what happens when we deal exclusively with recursive functions. His principal result is that, if $\{f_n(x)\}$ is a recursive enumeration of all primitive recursive functions, then $g(h(x, 0))$ is nonrecursive. *H. B. Curry* (State College, Pa.).

Skolem, Th. A note on recursive arithmetic. Norske Vid. Selsk. Forh. **17**, no. 27, 107–109 (1944). [MF 16891]

Skolem here returns to the theme of the note reviewed in the second preceding review. He applies the results of that note to prove in a simple way Kleene's theorem [reference in the third preceding review] that there is a primitive recursive relation $R(x, y)$ such that the class $(Ey)R(x, y)$ is nonrecursive. The relation $R(x, y)$ may even be of the form $y = f(x)$, where $f(x)$ is primitive recursive. *H. B. Curry.*

★**Daval, R., et Guilbaud, G.-T. Le Raisonnement Mathématique.** Presses Universitaires de France, Paris, 1945. 152 pp.

The brief analysis of the nature of mathematical reasoning contained in this book takes its start from a confrontation of Poincaré's discussions with the views of the logician Goblot. An extended presentation of mathematical induction, including forms of reasoning which do not immediately fall into this pattern, is followed by a consideration of the angle-sums of triangles and of polygons, in Euclidean and non-Euclidean geometry. The entire discussion is kept on an elementary level. Little account is taken of recent developments; no mention is made of intuitionism, although the role and value of intuition receive a considerable amount of attention. The following passage quoted from the preface to the second edition of Lebesgue's Leçons sur l'Intégration [Gauthier-Villars, Paris, 1928] suggests something of the temper and the quality of the book. La généralisation est l'un des meilleurs moyens de faire comprendre en mathématiques: alors que sur les cas particuliers on est gêné par tous les faits que l'on peut observer et qui ne sont propres qu'à ces cas particuliers, dans le cas général il n'y a plus rien à observer en dehors des faits mêmes sur lesquels il faut raisonner: on atteint ainsi au même résultat qu'avec les définitions axiomatiques. *A. Dresden.*

Destouches-Février, Paulette. Signification profonde du principe de décomposition spectrale. C. R. Acad. Sci. Paris **222**, 866–868 (1946). [MF 16286]

In a previous note [same C. R. **221**, 287–288 (1945); these Rev. **7**, 356] the author maintained that any physical theory containing the principle of spectral decomposition and involving quantities not simultaneously measurable is necessarily a subjectivist theory. She now argues that conversely any subjectivist physical theory necessarily involves the principle of spectral decomposition. The argument depends on the existence of a Hermitian form corresponding to an orthocomplemented modular lattice of subspaces of a Hilbert space. *O. Frink* (State College, Pa.).

Haantjes, J. The certainty of geometry. Euclides, Groningen **21**, 97–110 (1946). (Dutch)

Lecture at the Free University of Amsterdam.

Giovannini, Adriano. L'errore nelle matematiche. Period. Mat. (4) **22**, 57–65 (1942).

NUMBER THEORY

Brauer, Alfred. A theorem of M. Bauer. Duke Math. J. **13**, 235–238 (1946).

A generalization of a theorem of M. Bauer [J. Reine Angew. Math. **131**, 265–267 (1906)] is the author's theorem I. Let $f(x)$ be a polynomial with rational integral coefficients which has at least one real root. Let $G(k)$ be the group of the residue classes relatively prime to k and H a subgroup which does not contain the class of numbers congruent to $-1 \pmod{k}$. Then, for integral values of x, $f(x)$ takes values divisible by infinitely many primes which do not belong to the classes of H. The author obtains other results from this theorem. His proof is simple and such that the Dirichlet theorem is only essentially employed when the degree of $f(x)$ is even. *R. Hull* (Lincoln, Neb.).

Chowla, S. On quintic equations soluble by radicals. Math. Student **13**, 84 (1945).

Denote by $f(n)$ the number of quintic equations, having integral coefficients not exceeding n in absolute value, which can be solved by radicals. It is conjectured that $f(n)=o(n^6)$. Two theorems without proofs are given to support (but not prove) this conjecture. *I. Niven* (West Lafayette, Ind.).

Hukamchand. Proofs of some well known theorems in continued fractions. Math. Student **13**, 98–101 (1945).

Alternative proofs are given for three classic theorems concerning the continued fraction expansions of quadratic irrationals. *I. Niven* (West Lafayette, Ind.).

Ryde, Folke. Der Algorithmus der monotonen, nichtwachsenden Kettenbrüche. Ark. Mat. Astr. Fys. **31A**, no. 19, 18 pp. (1945).

The author considers the development of rational numbers $r>1$ in the form of monotonic nonincreasing continued fractions. These fractions are defined by

$$a+\cfrac{a}{b+\cfrac{b}{c+\cdots+\cfrac{k}{l+\cfrac{l}{1}}}},$$

where $a, b, c, \cdots, k, l$ are integers such that $a\geqq b\geqq c\geqq\cdots\geqq k\geqq l\geqq 1$. It is shown that certain rationals cannot be developed in this form, while others can be developed in more than one way. In particular, it is proved that there exists no r, $1<r<15/4$, which possesses more than one development in the form of a monotonic nonincreasing continued fraction. The author also derives an algorithm for obtaining developments. Placing $r=s_1+s_1/r_1$, we select for s_1 any one of the values $-[-\frac{1}{2}r]$, $-[-\frac{1}{2}r]+1$, $\cdots$, $[r-\frac{1}{2}]$, where $[a]$ denotes the greatest integer not greater than a, and where the set is assumed to contain at least one term. Similarly, we place $r_1=s_2+s_2/r_2$ and, in general, select for s_2 any one of the values $-[-\frac{1}{2}r_1]$, $-[-\frac{1}{2}r_1]+1$, $\cdots$, $[r_1-\frac{1}{2}]$. If, however, the value $[r-\frac{1}{2}]$ is chosen for s_1 and r satisfies certain inequalities, the set of integers from which s_2 is selected is subject to modifications. Proceeding in this way we obtain, in a finite number of steps, a denominator r_n which either equals 1 or satisfies one of the inequalities $1<r_n<3/2$, $2<r_n<5/2$. If $r_n=1$ the desired development is obtained; if r_n satisfies either inequality we may, perhaps, obtain it by selecting different values for s_1, s_2, etc. The paper contains a numerical table of developments of those rationals $r<15/4$ for which the sum of numerator and denominator is less than 10000. It concludes with the statement of certain congruences, modulo 2, and equalities, involving numerators and denominators of fractions expressed as monotonic nonincreasing continued fractions. *W. H. Gage* (Vancouver, B. C.).

Ryde, Folke. Über die rekursorische Berechnung der monotonen, nicht-wachsenden Kettenbrüche. Ark. Mat. Astr. Fys. **31B**, no. 12, 6 pp. (1945).

The author derives recursion formulas for the function $Nm(r)$, where $Nm(r)$ denotes the number of developments for a given rational number $r>1$ in the form of monotonic nonincreasing continued fractions [cf. the preceding review]. These formulas have been used to compute a table, which he intends to publish, for the function $Nm(r)$. *W. H. Gage* (Vancouver, B. C.).

Cabannes, Henri. Étude des fractions continues ayant leurs quotients en progression arithmétique ou en progression géométrique. Revue Sci. **83**, 230–233 (1945).

For a continued fraction whose quotients are in arithmetic progression, the numerator and denominator of the nth convergent are evaluated and consequently the continued fraction is expressed as a quotient of two series. This had been done by Euler, but (according to the author) without rigorous demonstration. Corresponding results are obtained for a continued fraction whose quotients are in geometric progression. The methods used are classical, involving properties of the binomial coefficients and the number of positive solutions of a general linear Diophantine equation. *I. Niven* (West Lafayette, Ind.).

Cabannes, Henri. Application des fractions continues à la formation de nombres transcendants. Revue Sci. (Rev. Rose Illus.) **82**, 365–367 (1944). [MF 16476]

Let $x_1=[\alpha_1;\alpha_2,\alpha_3,\cdots]$, the α_i positive integers; $p_n/q_n=[\alpha_1;\alpha_2,\alpha_3,\cdots,\alpha_n]$, $x_n=[\alpha_n;\alpha_{n+1},\alpha_{n+2},\cdots]$. Let $f_1(x)=a_0x^r+\cdots+a_r$, $f(x)$ irreducible in $R(1)$, $f_1(x_1)=0$. For $f_n(x)=x^rf_{n-1}(\alpha_{n-1}+1/x)$ $(n=2,3,\cdots)$ x_n is a simple root of $f_n(x)=0$. Several theorems are proved. (I) There exists an N such that for $n>N$ all roots of $f_n(x)=0$ except x_n have their real part $\Re(x)>-1$. (II) By the use of his inequality governing the approximation of algebraic numbers by rational numbers, Liouville was the first to construct sets of transcendental numbers. By an independent method, which is, however, related to the type of argument employed by Liouville, the author proves: if the positive integers $\alpha_1,\alpha_2,\alpha_3,\cdots$ of a number $[\alpha_1;\alpha_2,\alpha_3,\cdots]$ form an increasing sequence such that $\log\alpha_n/\log(\alpha_1\alpha_2\cdots\alpha_{n-1})$ increases indefinitely with n, the number is transcendental. (III) If the $\alpha_1,\alpha_2,\cdots$ of a number $[\alpha_1;\alpha_2,\alpha_3,\cdots]$ are such that $(\log\alpha_n)^{1/n}$ increases indefinitely with n, the number is transcendental. (IV) If $\log(1+\alpha_n)/\log\{(1+\alpha_1)\cdots(1+\alpha_{n-1})\}$ increases indefinitely with n, the number $[\alpha_1;\alpha_2,\alpha_3,\cdots]$ is transcendental. *A. J. Kempner* (Boulder, Colo.).

Uspensky, J. V. On a problem of John Bernoulli. I. Revista Unión Mat. Argentina **11**, 141–154 (1946). (Spanish)

John Bernoulli III (1744–1807) [Recueil pour les Astronomes, vol. 2, Berlin, 1772] considered the problem of constructing, for given positive x, the sequence of nearest integers to the multiples nx, $n=1,2,3,\cdots$, or, what is the same thing, the integers $[nx+\frac{1}{2}]$. The difference between consecutive terms is either $[x]+0$ or $[x]+1$ and we need only determine the sequence of zeros and ones. The prob-

lem was studied by A. Markoff [Math. Ann. 19, 27–36 (1882)]. One may consider $[nx+b]$, for given x and b, $0<x<1$. The corresponding sequence of zeros and ones is denoted by (x, b). If $x=P/Q$ is an irreducible rational fraction, then the sequence (x, b) is periodic, with Q terms in its period; the period contains P units, $Q-P$ zeros; the sequence $(P/Q, b)$ is the same as $(P/Q, r/Q)$ for a certain integer r; the period of $(P/Q, r/Q)$ is a cyclic permutation of the period $c_0, \cdots, c_{Q-1}$ of $(P/Q, 0)$; $c_1, \cdots, c_{Q-1}$ is symmetric. The period of $(P/Q, 0)$ can be written down by applying simple rules to the continued fraction expansion of P/Q. Whether x is rational or irrational, a method of finding the first N terms can be used. If P/Q is a convergent of even order of the continued fraction for x, so that $P/Q=x+\epsilon/Q^2$, $0<\epsilon<1$, if ω is the maximum of the fractional parts of $nx+b$, $n=0, \cdots, N$, and Q is chosen to satisfy $\omega+N\epsilon/Q^2<1$, then $[mP/Q+b]=[mx+b]$, $m=0, \cdots, N$. Hence the problem reduces to that for the sequence $(P/Q, (2r+1)/(2Q))$ for some integer r. *G. Pall.*

Uspensky, J. V. On a problem of John Bernoulli. II. Revista Unión Mat. Argentina 11, 165–183 (1946). (Spanish)

Continuing the article reviewed above, several theorems are proved. The minimal period of a section consisting of N terms of a sequence (x, b) is the period of a certain Bernoulli sequence $(p/q, (2s+1)/(2q))$. If $(p/q, s/q)$ generates the minimal period of the section $K_0, \cdots, K_{q-1}$ of the sequence (x, b), then p/q is a principal or intermediate convergent to the continued fraction for x and is a best approximation. If p/q is a best approximation and the sequence $(p/q, \nu/q)$ has g terms in common with (x, b), then $g \geqq q$ and the same sequence generates a minimal period of the section $K_0, \cdots, K_{q-1}$. *G. Pall* (Chicago, Ill.).

Vinogradow, I. M. A general distribution law for the fractional parts of values of a polynomial with the variable running over the primes. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 491–492 (1946).

Estimates for trigonometric sums such as

$$\sum_{k=1}^{K} \left| \sum \exp\left(2\pi i k f(p)\right) \right|,$$

where K is a positive constant, $f(p)=a_n p^n+a_{n-1}p^{n-1}+\cdots+a_1 p$, a_i real, and the inner sum is over all primes $p \leqq P$, have previously been made to depend on the approximation of the leading coefficient a_n by rational numbers a/q. In this paper an estimate is given which depends on the approximation of any coefficient of the polynomial by rational numbers. The new estimate is then applied to the problem indicated in the title. The main result is the following asymptotic formula for the number T of fractional parts $\{f(p)\}$ satisfying $0 \leqq \{f(p)\} \leqq \sigma$, $p \leqq P$:

$$T=\sigma\pi(P)+O(P^{1+\epsilon}(q^{-1}+P^{-1})^{\rho}),$$

where $a_r=a/q+z$; r is one of the numbers $1, 2, \cdots, n$; $|z| \leqq 1/q\tau$; $\tau=P^{r/2}$; $\rho=1/(8n^2 \log 12n)$. No proofs are given. *R. D. James* (Vancouver, B. C.).

Rosenblatt, Alfred. On the natural diatonic scales. I, II. Actas Acad. Ci. Lima 8, 165–182, 183–196 (1945). (Spanish)

The tones in an octave of the natural diatonic scale correspond to the numbers $1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8, 2$. It is easy to express these in terms of the numbers $\alpha=9/8$, $\beta=10/9$, $\gamma=16/15$, which correspond to a "major tone," a "minor tone" and a "diatonic semi-tone," respectively. The numbers corresponding to all the natural tones of all the natural ascending scales (in all octaves) form a multiplicative set G_1. It is shown that this set is uniformly dense in the sense of Weyl and various sets of generators of G_1 are given. The tones generated by those corresponding to α, β, γ in all the octaves are also considered; the numbers corresponding to these form a set G_2 which contains G_1 as a proper subset. The characterization of G_1 as a subset of G_2 is given in a simple manner. *H. W. Brinkmann.*

Griffiths, L. W. A note on linear homogeneous Diophantine equations. Bull. Amer. Math. Soc. 52, 734–736 (1946).

The solution of the Diophantine system

$$(1) \qquad a_{i1}x_1+\cdots+a_{in}x_n=0, \qquad i=1, \cdots, m,$$

of rank r is well known if $r=m=n-1$ ($x_j=(-1)^j t E_j/e$, where t is an arbitrary integer, E_j is the jth minor of order $n-1$ and $e=(E_1, \cdots, E_j)$). The author proves that, if $r=m<n-1$, the complete solution can be obtained by applying this to the system formed by (1) and the equations $\xi_{i1}x_1+\cdots+\xi_{in}x_n=0$, $i=1, \cdots, n-m-1$, where the ξ_{ij} are arbitrary integer parameters. [Reviewer's note. A simpler proof of this result may be obtained by application of its analogue in the rational number field.] *N. G. de Bruijn.*

Kraitchik, Maurice. On certain rational cuboids. Scripta Math. 11, 317–326 (1945).

The Diophantine system $x^2+y^2=Z^2, y^2+z^2=X^2, z^2+x^2=Y^2$ is discussed. Among other results, Euler's formulas [see L. E. Dickson, History of the Theory of Numbers, vol. 2, Washington, 1920, p. 498] are proved. A table of 50 solutions not derivable from Euler's formulas is given. *I. Niven* (West Lafayette, Ind.).

Ljunggren, Wilhelm. Zur Theorie der Gleichung $x^2+1=Dy^4$. Avh. Norske Vid. Akad. Oslo. I. 1942, no. 5, 27 pp. (1942). [MF 16434]

It is proved that, if the fundamental unit of the real quadratic field $R(\sqrt{D})$ is not also the fundamental unit of the ring of integers of the field, then $x^2+1=Dy^4$ has at most two solutions in positive integers. It is shown how the possible solutions may be found. The result is best possible: for $D=2$ the solutions are $x=1$, $y=1$ and $x=239$, $y=13$. The proof depends on earlier work of the author [Skr. Norske Vid. Akad. Oslo. I. 1936, no. 12] on the units of quadratic and biquadratic fields. *I. Niven.*

Mebius, C. A. Zahlentheoretische Untersuchungen. III. Die Diophantische Gleichung $A^3+B^3-C^3-D^3=E$. Göteborgs Kungl. Vetenskaps- och Vitterhets-Samhälles Handlingar (6) Ser. B. 3, no. 6, 21 pp. (1945). [MF 16954]

By use of the identity $A^3+B^3-C^3-D^3=(12abc)^3E$, where $A, B=6a^3bc \pm b(3c^3+e)$, $C, D=6ab^3c \pm a(3c^3-e)$, $e=4E+b^6-a^6$, the author proves that every integer of the form $3n$ is a sum of 4 integral cubes, positive or negative [a result due to Richmond, Messenger of Math. 51, 177–186 (1921)], and that any integer expressible as a sum of 4 cubes is so expressible in infinitely many ways. The identity is general in the sense that any integer expressed as a sum of 4 cubes can be treated by appropriate choice of a, b, c, not always rational or even real; many of the classical results are thus obtained. Several special cases of the identity are discussed. *I. Niven* (West Lafayette, Ind.).

Skolem, Th. On certain exponential equations. Norske Vid. Selsk. Forh. **18**, no. 18, 71–74 (1945). [MF 16893]

The Diophantine equation $a^x-1=p_1^{x_1}\cdots p_m^{x_m}$ (a a given integer, $p_1, \cdots, p_m$ given primes) is proved to have at most a finite number of solutions in nonnegative integers $x, x_1, \cdots, x_m$. This is a special case of a theorem of Størmer. The method used here can be extended to algebraic number fields. The following theorem is obtained. If α is a given algebraic integer, no conjugate of which has the absolute value 1, and $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$ are given prime ideals in a field containing α, then an upper bound can be found for the natural numbers x with the property that the decomposition of α^x-1 contains only $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$. *N. G. de Bruijn.*

Skolem, Th. On the prime divisors of the values of certain functions. Norske Vid. Selsk. Forh. **18**, no. 19, 75–78 (1945). [MF 16892]

Generalizing a theorem concerning the prime divisors of a^x-1 [cf. the preceding review] the author proves the following theorem. Let $f(x)$ attain rational integral values for all rational integers x, let $p_1, \cdots, p_m$ denote different primes and let $r_1, \cdots, r_l$ be l integers. For $j=1, \cdots, m$; $n=1, 2, 3, \cdots$, suppose that $f(x)\equiv 0 \pmod{p_j^n}$ implies that x belongs to one of the l residue classes $x\equiv r_i \pmod{\psi_j(n)}$. Now if f and ψ_j satisfy

$$\liminf_{x\to\pm\infty} (\log |f(x)|)/\log |x| > m,$$

$$\liminf_{n\to\infty} n^{-1}\log \psi_j(n) \geqq \log p_j, \qquad j=1, \cdots, m,$$

then the equation $f(x)=kp_1^{x_1}\cdots p_m^{x_m}$ has, for a given value of k, at most a finite number of solutions in rational integers $x, x_1, \cdots, x_m$. A second theorem deals with the same equation under slightly different conditions.

N. G. de Bruijn (Eindhoven).

Skolem, Th. A method for the solution of the exponential equation $A_1^{x_1}\cdots A_m^{x_m}-B_1^{y_1}\cdots B_n^{y_n}=C$. Norsk. Mat. Tidsskr. **27**, 37–51 (1945). (Norwegian) [MF 16886]

Using a lemma on the Pell equation, Størmer showed [Nyt Tidsskr. Math. B. **19**, 1–7 (1908)] how to obtain all integral solutions x_i, y_j of the exponential equation of the title when $C=\pm 1$ or ± 2 and A_i, B_j are given integers. The author treats the general case $|C|\geqq 3$ and shows how it is possible to solve the more general problem: find all integral solutions X, Y of the Diophantine equation $X^3-DY^3=E$, when D, E and the prime divisors of Y are all given. There is only a finite number of solutions. In his proof the author uses the properties of p-adic numbers and of the cubic field generated by $D^{\frac{1}{3}}$. In the case $C=\pm 3$ the proof is not complete. *T. Nagell* (Uppsala).

Skolem, Th. Unlösbarkeit von Gleichungen, deren entsprechende Kongruenz für jeden Modul lösbar ist. Avh. Norske Vid. Akad. Oslo. I. **1942**, no. 4, 28 pp. (1942). [MF 16433]

It is known that linear homogeneous quadratic Diophantine equations have solutions if the corresponding congruences are solvable for every modulus. This is, however, not correct for inhomogeneous quadratic equations and equations of higher degree. The author shows that, for instance, $(x^2-2)(x^2+7)(x^2+14)\equiv 0 \pmod m$ is solvable for every m while the corresponding equation has no rational root.

Let $f(x, y)$ be a cubic form. In an earlier paper [same Avh. **1937**, no. 12] it was mentioned by the author that the equation $f(x, y)=$ constant may have no solution while the corresponding congruence is solvable for every modulus m. He gave the example $x^3-2y^3=47$. He now states that this example is not correct since $x=63$, $y=50$ is a solution. In order to obtain correct examples he proves the following theorems. Let p be a prime of form $3n+2$ and $(a, b)=(a, c)=(b, c)=1$; then $ax^3+by^3\equiv c \pmod{p^n}$ has solutions for every n. Moreover, this congruence has solutions if $p\neq 7$ is an arbitrary prime and $3abc\not\equiv 0 \pmod p$. Using these results the author proves that $x^3+3y^3=22$ and $u^3+4v^3=55$ have no solutions while the corresponding congruences are solvable for every m. Moreover, he proves criteria for the solvability of congruences of the form $N(\alpha_1 x+\alpha_2 y)\equiv 1 \pmod{p^n}$, where α_1 and α_2 are integers of a cubic field.

A. Brauer (Chapel Hill, N. C.).

Skolem, Th. A relation between the congruence $x^2+y^2+z^2+u^2\equiv 0 \pmod m$ and the equation $x^2+y^2+z^2+u^2=m$. Norsk. Mat. Tidsskr. **25**, 76–87 (1943). (Norwegian) [MF 16885]

The author gives a new proof of the theorem of Jacobi on the number of representations of a positive integer m as a sum of four squares. A proof by Hurwitz [Nachr. Ges. Wiss. Göttingen. Math. Phys. Kl. **1896**, 314–340] is based on the theory of quaternions. The author's proof, which is much simpler, also uses the arithmetic of quaternions, but only its elements. It makes evident the relation between the equation $x^2+y^2+z^2+u^2=m$ and the congruence $x^2+y^2+z^2+u^2\equiv 0 \pmod m$. *T. Nagell.*

Skolem, Th. A theorem on the equation $\zeta^2-\delta\eta^2=1$ where δ, ζ, η are integers in an imaginary quadratic field. Avh. Norske Vid. Akad. Oslo. I. **1945**, no. 1, 13 pp. (1945). [MF 16441]

Let k be an imaginary quadratic field different from $R(\sqrt{(-1)})$ and $R(\sqrt{(-3)})$ and let δ be any integer in k not having any of the forms τ^2, $-\tau^2$, $-3\tau^2$, for τ in k. Say that a solution of $\zeta^2-\delta\eta^2=1$ in k has property P when all prime ideals which divide η also divide δ. It is proved that the number of essentially different solutions which have property P never exceeds two. If there is only one such solution it is α, β, where $\alpha+\beta\sqrt{\delta}$ is a fundamental unit for the group of all units in the field $k(\sqrt{\delta})$ having the relative norm $+1$. If there are two solutions they are α, β and α_3, β_3, determined by $\alpha_3+\beta_3\sqrt{\delta}=(\alpha+\beta\sqrt{\delta})^3$. Furthermore, only the solution α, β can have property P if 3 is a prime number in k or the square of a prime ideal in k. Necessary and sufficient conditions for only α, β to have property P are given in case 3 is the product of two different prime ideals in k.

I. Niven (West Lafayette, Ind.).

Skolem, Th. A remark on the equation $\zeta^2-\delta\eta^2=1$, $\delta>0$, $\delta', \delta'', \cdots<0$, where δ, ζ, η belong to a total real number field. Avh. Norske Vid. Akad. Oslo. I. **1945**, no. 12, 15 pp. (1945). [MF 16882]

Let k be an algebraic number field of degree n which contains a totally real element δ such that $\delta>0$; $\delta', \delta'', \cdots, \delta^{(n-1)}<0$; then δ is necessarily primitive in k and $[k(\sqrt{\delta}):k]=2$. The group of units of $k(\sqrt{\delta})$ having norm $+1$ relative to k is cyclic, as is the subgroup of such units of the form $\zeta+\eta\sqrt{\delta}$, ζ, η integers in k, and hence the solutions of the equation $\zeta^2-\delta\eta^2=1$ may be obtained from the powers of the unit which generates this subgroup. The author shows that there is only a finite number of solutions of this equation such that every prime ideal divisor of η

also divides δ, that an upper bound to the number of these solutions may always be found, and that in certain cases all such solutions may be effectively obtained. As a consequence of these results he deduces the following theorem. Let $A, A_1, \cdots, A_m, B, B_1, \cdots, B_n$ be integers in a totally real algebraic number field K with $AB>0$, $A'B'<0$, $A''B''<0$, $\cdots$; $A_1, \cdots, A_m, B_1, \cdots, B_n$ totally positive. Then the equation $AA_1^{x_1}\cdots A_m^{x_m}-BB_1^{y_1}\cdots B_n^{y_n}=1$ has only a finite number of solutions in rational integers $x_1, \cdots, x_m, y_1, \cdots, y_n$. If none of A_i or B_j is a unit in K, then all solutions can be obtained, while in general an upper bound to the number of solutions may be found. Similar theorems are shown to hold in any field which is quadratic, cyclic cubic or cubic with negative discriminant.

S. A. Jennings (Vancouver, B. C.).

Skolem, Th. Extension of two theorems of C. Størmer. Norsk. Mat. Tidsskr. **26**, 85–95 (1944). (Norwegian) [MF 16887]

Let α, β, δ be integers in a totally real field K of algebraic numbers, α and $\beta\neq 0$, δ totally positive, prime to α and to the discriminant of the field K and not a square in K. The author shows that, if $A=\alpha+\beta\sqrt{\delta}$, $A^x=\alpha_x+\beta_x\sqrt{\delta}$, x a positive integer, and if every prime ideal factor of β_x divides δ, then $x=1$, 2 or 3.

This result is a generalization of a theorem of Størmer [Skr. Norske Vid.-Selsk. Christiania. I. **1897**, no. 2] in the rational field R. The author conjectures that an analogous result holds for any field without restrictions; a proof is given only for an imaginary quadratic field. A consequence of the results is an extension to algebraic fields of a theorem of Størmer on the solutions of a certain type of exponential Diophantine equation in the rational field R [C. R. Acad. Sci. Paris **127**, 752–754 (1898); Nyt Tidsskr. Mat. B. **19**, 1–7 (1908)]. The number of solutions is finite; the solutions may all be determined by a finite number of operations.

T. Nagell (Uppsala).

Gut, Max. Zur Theorie der Normenreste einer relativ-zyklischen Erweiterung von ungeradem Primzahlgrade. Vierteljschr. Naturforsch. Ges. Zürich **91**, 17–36 (1946).

The author considers cyclic extensions K, of odd prime degree l, of an algebraic number field k. For each power of a prime ideal in k he determines the index of norms mod norm residues. He proves also that, if $\mathfrak{I}=\mathfrak{L}^l$, where $\mathfrak{l}$ and $\mathfrak{L}$ are ideals in k and K, respectively, and if $N_{K/k}\Theta\equiv 1 \bmod$ a power of $\mathfrak{I}$ at least as high as that dividing the conductor, then $\Theta\equiv\Delta^{1-S} \bmod$ a certain power of $\mathfrak{L}$, where S is an automorphism generating the Galois group. These results are known [H. Hasse's report on class field theory, Jber. Deutsch. Math.-Verein. **36**, 233–311 (1927)]. The proofs given here are more elementary and explicit, but somewhat longer, than former ones. No p-adic theory is used.

G. Whaples (Madison, Wis.).

Moriya, Mikao. Über die Struktur der Divisorenklassen einer zyklischen Erweiterung von Primzahlgrad über einem algebraischen Funktionenkörper. Tôhoku Math. J. **48**, 43–54 (1941). [MF 16345]

Consider the fields $k_0\subset k\subset K\subset Z$, where k_0 is a finite field, k is an infinite algebraic extension of k_0, K is an algebraic function field of one variable with k as constant field and Z is a cyclic separable extension of K of prime degree l. Assume that k has no algebraic extension of degree l. The author shows that the number N of divisor classes C in Z which have $C^l=1$ can be determined as $N=l^a$, where $a=\rho l+(t-1-b)(l-1)$ and (i) ρ is determined from the number l^ρ of divisor classes C of the same sort in K, (ii) t is the number of prime divisors of K which are ramified in Z, (iii) $b=1$ if K contains the primitive l-roots of unity and $b=0$ otherwise. In the special case when k is algebraically closed, this result was obtained by Deuring [Math. Ann. **113**, 208–215 (1936)]. The proof depends essentially on the fact that each divisor class C of the indicated sort can be obtained from a divisor class C_0 of an approximating field Z_0/K_0, in which the coefficient field is finite. For fields of the latter sort, the class field theory for algebraic function fields with finite coefficient fields [see, for example, Moriya, Jap. J. Math. **14**, 67–84 (1938)] can be effectively applied.

S. MacLane (Cambridge, Mass.).

***de Bruijn, Nicolaas Govert. Over Modulaire Vormen van Meer Veranderlijken.** [**On Modular Forms in Several Variables**]. Thesis, Free University of Amsterdam, 1943. xvi+63 pp. (Dutch)

Modular functions of several variables were first associated with algebraic fields by Hilbert and by Blumenthal [Math. Ann. **56**, 509–548 (1903); **58**, 497–527 (1904)]. The purpose of the present paper is to extend Blumenthal's results in the direction which Hecke and his associates have followed in the theory of modular functions of one variable.

Let K be a totally real algebraic field of degree n. The matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ with $\alpha, \beta, \gamma, \delta\in K$ and $\alpha\delta-\beta\gamma=\sigma\neq 0$ gives rise to a set of substitutions

$$\tau_1^{(i)}=\frac{\alpha^{(i)}\tau^{(i)}+\beta^{(i)}}{\gamma^{(i)}\tau^{(i)}+\delta^{(i)}}, \qquad i=1, \cdots, n, \tag{1}$$

where $\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, \delta^{(i)}$ are the conjugates of $\alpha, \beta, \gamma, \delta$ and $\tau^{(i)}$ are complex variables. Blumenthal studied the group H of substitutions with $\alpha, \beta, \gamma, \delta$ algebraic integers and σ a totally positive unit.

A modular form of dimension $-k$ is defined as a function of n variables $\tau^{(1)}, \cdots, \tau^{(n)}$, subject to certain regularity conditions and with the property that

$$F\left(\frac{\alpha^{(1)}\tau^{(1)}+\beta^{(1)}}{\gamma^{(1)}\tau^{(1)}+\delta^{(1)}}, \cdots, \frac{\alpha^{(n)}\tau^{(n)}+\beta^{(n)}}{\gamma^{(n)}\tau^{(n)}+\delta^{(n)}}\right) = \prod_{i=1}^{n}(\gamma^{(i)}\tau^{(i)}+\delta^{(i)})^k F(\tau^{(1)}, \cdots, \tau^{(n)}) \tag{2}$$

in the region $\Im(\tau^{(i)})>0$. Instead of the Hilbert group H a wider group Γ_0 (already mentioned by Hurwitz) is studied which is defined as follows. Let $\mathfrak{e}$ be such an ideal of K (not necessarily entire) that $\mathfrak{e}^2$ is a principal ideal; then the substitution with the matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ belongs to Γ_0 if $\alpha, \beta, \gamma, \delta$ lie in $\mathfrak{e}$ and $\sigma=\alpha\delta-\beta\gamma$ fulfills $(\sigma)=\mathfrak{e}^2$. Hilbert's group H is a subgroup of Γ_0, and so is the group Γ_1, with $\alpha, \beta, \gamma, \delta$ entire and $\sigma=1$.

For Γ_0 the determinant σ does not need to be positive. The region $\Im(\tau^{(i)})>0$ is then transformed through (1) into $\sigma^{(i)}\Im(\tau^{(i)})>0$. Therefore regions T_w are introduced, where $w=(w^{(1)}, \cdots, w^{(n)})$ is a signature, $w^{(i)}$ being ± 1, and T_w is determined by $w^{(i)}\Im(\tau^{(i)})>0$, $i=1, \cdots, n$. If $M=\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, the author defines (following Kloosterman)

$$F(\tau)\,|\,M=N(\gamma\tau+\delta)^{-k}F\left(\frac{(\alpha\tau+\beta)}{(\gamma\tau+\delta)}\right),$$

where $F(\tau)$ is an abbreviation for $F(\tau^{(1)}, \cdots, \tau^{(n)})$. A modular form of type $(\Gamma_1, -k)$ is now defined as a function $F(\tau)$ which satisfies the conditions (1) $F(\tau)\,|\,L=F(\tau)$ for all $L\in\Gamma_1$,

(2) $F(\tau)$ is regular in all T_w, (3) for every matrix $M=\binom{\alpha\ \beta}{\gamma\ \delta}$ with determinant $\sigma\neq 0$ there exists an ideal $\mathfrak{m}$ such that for any signature w and $\tau\varepsilon T_w$ there exists an absolutely convergent expansion

$$(3)\qquad F(\tau)\,|\,M=c(0)+\sum_{w\nu>0,\ \nu\equiv 0(\mathfrak{m})} c(\nu)\exp 2\pi i S(\nu\tau).$$

Condition (3) refers to the "cusps" of the fundamental region in which regularity is demanded.

A still finer classification of modular forms is obtained by the use of the wider group Γ_0 instead of Γ_1 in the following way. The subgroup $\Gamma_{(\mu)}$ of those substitutions L of Γ_0 for which the ideal $\mathfrak{e}$ is a principal ideal (μ) with $N(\mu)^k>0$ is normal and of finite index in Γ_0. Let χ be a character of the factor group $\Gamma_0/\Gamma_{(\mu)}$. Then the form F is said to be of type $(\Gamma_0, -k, \chi)$ if conditions (1), (2), (3) are fulfilled and if also, for any substitution $M=\binom{\alpha\ \beta}{\gamma\ \delta}$ of Γ_0, $\alpha\equiv\beta\equiv\gamma\equiv\delta\equiv 0 \pmod{\mathfrak{e}}$, $(\sigma)=\mathfrak{e}^2$, the equation $N(\mathfrak{e})^k F\,|\,M=\chi(M)F$ is satisfied. Any function of type $(\Gamma_1, -k)$ can then be written as a sum of functions of type $(\Gamma_0, -k, \chi)$, the sum extended over all χ.

After this preparation, the analogues of Hecke's operators $T(n)$ can be introduced. First it can be seen that χ can be defined for all "singular numbers" σ, that is, those whose principal ideal is a square, by just putting $\chi(\sigma)=\chi(M)$, where σ is the determinant of M; second, χ can then be extended to all numbers, not equal to 0, of K. Let now $\mathfrak{n}$ be an entire ideal which is equivalent to the square of an ideal. The operator $T(\mathfrak{n}, \chi)$ is then defined by

$$(4)\qquad F\,|\,T(\mathfrak{n}, \chi)=N(\mathfrak{n})^{k-1}N(\mathfrak{b})^k\bar{\chi}(\sigma)\sum_{i=1}^{s} F\,|\,M_i.$$

Here $M_1, \cdots, M_s$ is a complex system of left classes modulo Γ_1 of matrices "of order $\mathfrak{n}$ with determinant σ," that is, with $\alpha\delta-\beta\gamma=\sigma$, $(\sigma)=\mathfrak{n}\mathfrak{b}^2$, $\alpha\equiv\beta\equiv\gamma\equiv\delta\equiv 0 \pmod{\mathfrak{b}}$. The definition (4) does not depend on the choice of σ.

These operators are commutative as in Hecke's case and the author obtains, for two entire ideals $\mathfrak{N}$ and $\mathfrak{n}$ which are equivalent with ideal squares, for each form F of type $(\Gamma_0, -k, \chi)$, the formula

$$F\,|\,T(\mathfrak{N}, \chi)T(\mathfrak{n}, \chi)=\sum_{\mathfrak{c}\,|\,(\mathfrak{N}, \mathfrak{n})} N(\mathfrak{c})^{k-1}F\,|\,T(\mathfrak{N}\mathfrak{n}/\mathfrak{c}^2, \chi),$$

in full analogy to Hecke's formula. Parallel with this goes, as in Hecke's theory, a relation between the coefficients of expansions in cusp points:

$$c(\mathfrak{n}, F\,|\,T(\mathfrak{N}, \chi))=\sum_{\mathfrak{c}\,|\,(\mathfrak{N}, \mathfrak{n})} N(\mathfrak{c})^{k-1}c(\mathfrak{N}\mathfrak{n}/\mathfrak{c}^2, F),$$

where the $c(\mathfrak{n}, F)$ are defined on the basis of the expansion (3).

An analogue for Hecke's representation of the operators $T(n)$ by matrices $\lambda(n)$ is still missing since the finiteness of a basis of the linear family of forms of given dimension is a function-theoretical problem which the author leaves untouched. However, for eigenfunctions F of the operators $T(\mathfrak{n}, \chi)$, that is, for functions which fulfill $F\,|\,T(\mathfrak{n}, \chi)=\lambda(\mathfrak{n})F$ with complex numbers $\lambda(\mathfrak{n})$, these form a representation of the ring of the $T(\mathfrak{n}, \chi)$. Eigenfunctions can indeed be constructed by means of Eisenstein's series [Kloosterman, Abh. Math. Sem. Hamburgischen Univ. 6, 163–188 (1928)].

The second part of the paper applies some of the methods to "special" ϑ-series in n variables. Let N be an even number and let $\xi=(\xi_1, \cdots, \xi_N)$ be a vector whose components are numbers of K. The quadratic form $Q(\xi, \xi)=\sum_{i,j=1}^{N}\alpha_{ij}\xi_i\xi_j$ is called totally definite if there exists a signature w such that $w^{(i)}Q(\xi^{(i)}, \xi^{(i)})>0$ for all ξ; w is then called the signature of Q. If $\mathfrak{b}$ is an ideal and $\rho=(\rho_1, \cdots, \rho_N)$ a vector, $\rho_j\varepsilon K$, and Q such a form that $\alpha_{ij}\varepsilon\mathfrak{b}^{-2}\mathfrak{d}^{-1}$, $\alpha_{ii}\varepsilon 2\mathfrak{b}^{-2}\mathfrak{d}^{-1}$,

$$Q(\rho)=\sum_{j=1}^{N}\alpha_{ij}\rho_j\varepsilon\,\mathfrak{b}^{-1}\mathfrak{d}^{-1},$$

with $\mathfrak{d}$ the "differente" of K, then

$$\theta(\tau, Q, \rho, \mathfrak{b})=\sum_{\xi\equiv\rho(\mathrm{mod}\ \mathfrak{b})}\exp \pi i S(\tau Q(\xi, \xi))$$

is introduced as a "special ϑ-series." For the sake of convergence, $\tau\varepsilon T_{\text{sign }Q}$. The series θ is of a certain level (Stufe) $\mathfrak{M}$, depending on the quadratic form Q, that is, for the group $\Gamma_1(\mathfrak{M})$ of all those matrices L for which

$$L\equiv\binom{1\ \ 0}{0\ \ 1} \bmod \mathfrak{M},$$

we have $\theta\,|\,L=\theta$. The author finally constructs special ϑ-series of level 1 by means of a known definite quadratic form in the rational field of 8 variables and determinant 1, which goes back to Korkine and Zolotareff and which has recently been studied again by Mordell [J. Math. Pures Appl. (9) 17, 47–87 (1938)]. *H. Rademacher.*

de Bruijn, N. G. Almost periodic multiplicative functions. Nieuw Arch. Wiskunde (2) 22, 81–95 (1943). (Dutch) [MF 15701]

The author establishes necessary and sufficient conditions for an arithmetic multiplicative function to be almost periodic in the sense of Bohr. *M. Kac* (Ithaca, N. Y.).

Wintner, Aurel. Square root estimates of arithmetical sum functions. Duke Math. J. 13, 185–193 (1946).

In Gauss's circle problem it can be proved by elementary methods that $\sum_{m=1}^{N}f(m)=\pi n+O(n^{\frac{1}{2}})$, where $f(m)$ is the number of representations of m as a sum of two squares. More generally, the author shows that, for an arbitrary function $f(m)$,

$$(A)\qquad \sum_{m=1}^{n} f(m)=n\sum_{m=1}^{\infty} m^{-1}f'(m)+O(n^{\frac{1}{2}})$$

provided that (B) $\sum_{m=1}^{n}f'(m)=O(1)$, where $f(n)=\sum_{d|n}f'(d)$. The exponent $\theta=\frac{1}{2}$ in the error term in (A) can, of course, be improved in special cases (for example, in the circle problem). The author proves, however, that in the general case nothing more than (A) can be concluded from (B); that is, functions $f(m)$ exist for which the error term cannot be replaced by one of order lower than $n^{\frac{1}{2}}$. Thus "the possibility of improving the Gauss-Dirichlet exponent $\theta=\frac{1}{2}$ is due to the explicit nature of their problem" and not to "a general arithmetical fact depending only on that peculiarity of the sequence 1, 2, $\cdots$ which makes possible the application of the underlying sieve process." Use is made of the Lebesgue-Toeplitz "norm" principle [Toeplitz, Prace Mat.-Fiz. 22, 113–119 (1913)]. A similar result is shown to hold for problems of the type of Dirichlet's divisor problem.

The reviewer observes that the inequality following (13) and the last paragraph on page 188 are incorrect but can be modified so as to leave the final conclusions unaltered. Furthermore, on page 187, line 7 and elsewhere the symbol $O(n^{\frac{1}{2}})$ is used in a more restricted sense than is customary. There are also several minor misprints. *R. A. Rankin.*

Romanoff, N. P. Hilbert space and the theory of numbers. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 3–34 (1946). (Russian. English summary)

A sequence $\{f_r\}$ $(r=1, 2, \cdots)$ of elements of a Hilbert space H is called a D_σ-sequence if the scalar product

$(f_m, f_n) = g\{(m, n)\}$ for all m, n, where $g(r)$ is a given function of the positive integral variable r, and (m, n) denotes the highest common factor of m and n. The author shows how D_g-sequences can be constructed from any given orthonormal sequence in H. Thus, if $\{\alpha_r\}$ is an orthonormal sequence, and $\omega(n)$ is any complex-valued function satisfying $\omega(ab) = \omega(a)\omega(b)$, $\omega(a) \neq 0$ for all positive integral a, b, and $\sigma = \sum_{k=1}^{\infty} |\omega(k)|^2 < \infty$, then $\{f_n\} = \{\overline{\omega(n)}^{-1} \sum_{k=1}^{\infty} \omega(k)\alpha_{kn}\}$ is a D_g-sequence for $g(n) = \sigma|\omega(n)|^{-2}$, and a second orthonormal sequence $\{\psi_n\}$ is given by

$$\psi_n = \{\sigma\Omega(n)\}^{-1} \sum_{d|n} \mu(n/d) f_d,$$

where $\Omega(n) = \sum_{d|n} \mu(n/d)|\omega(d)|^{-2}$ and $\mu(n)$ is the Möbius function. Furthermore, if $\{\alpha_n\}$ is complete in a subspace H' of H then so is $\{\psi_n\}$. The transition from $\{\alpha_n\}$ to $\{\psi_n\}$ defines a transformation of matrix $C_\omega = \|c_{ik}\|$, where

$$c_{ik} = (\psi_i, \alpha_k) = \frac{\omega(k)}{\{\sigma\Omega(i)\}^{\frac{1}{2}}} \sum_{d|(i,k)} \mu(i/d)|\omega(d)|^{-2}.$$

From the known properties of orthonormal sequences in Hilbert space it can be concluded that C_ω is unitary, but the author gives, in addition, two direct proofs, the second of which arises as a particular case of a discussion of the form of the matrix $T = \|t_{ik}\| = C_{\omega_1}^* C_\omega$, where $\omega(n)$ and $\omega_1(n)$ are multiplicative functions of the type mentioned above. Complicated expressions for t_{ik} are derived in terms of infinite products over the primes. Similar results are also derived for biorthogonal sequences and for functions $\omega(n)$ which may vanish for certain values of n.

The two alternative representations $(F, G) = \sum_{k=1}^{\infty} a_k \bar{b}_k = \sum_{n=1}^{\infty} (F, \psi_n)\overline{(G, \psi_n)}$ of the scalar product of two elements $F = \sum a_k \alpha_k$ and $G = \sum b_k \alpha_k$ ($\sum |a_k|^2 < \infty$, $\sum |b_k|^2 < \infty$) in the space lead, for the choice $\omega(n) = n^{-1/2}$ ($s > 1$), to the identity

$$(*) \qquad \zeta(s) \sum_{k=1}^{\infty} a_k \bar{b}_k = \sum_{n=1}^{\infty} \{\Omega(n)\}^{-1} A_n \bar{B}_n,$$

where

$$A_n = \sum_{d|n} \mu(n/d) d^{1/2} \sum_{k=1}^{\infty} k^{-1/2} a_{kd}$$

and B_n is defined similarly. Several particular cases of (*) are given for different choices of a_k, b_k. A second paper is to be devoted to the application of the methods developed to the theory of the distribution of primes and to other questions. *R. A. Rankin* (Cambridge, England).

Ramanathan, K. G. Congruence properties of Ramanujan's function $\tau(n)$. II. J. Indian Math. Soc. (N.S.) **9**, 55–59 (1945).

[For part I see Proc. Indian Acad. Sci., Sect. A. **19**, 146–148 (1944); these Rev. **6**, 37.] Congruences are given for Ramanujan's function $\tau(n)$ taken with respect to the moduli 3, 4 and 7. All three theorems of the paper may be combined as follows. Let k be 3, 4 or 7 and let n be any positive integer. If k is not prime to n then $\tau(n)$ is divisible by k. If k is prime to n we may write $n = PQ$, where Q is divisible only by primes which are quadratic residues of k. Then if P is not a square, $\tau(n)$ is divisible by k; if P is a square, $\tau(n)$ is congruent modulo k to the number of divisors of Q. For $k = 7$ this result is due to J. R. Wilton [Proc. London Math. Soc. (2) **31**, 1–10 (1930)]. The paper uses elementary methods based on identities between familiar elliptic modular functions. *D. H. Lehmer* (Berkeley, Calif.).

Lahiri, Debabrata. Congruence properties of Ramanujan's function $\tau(n)$. Science and Culture **12**, 52 (1946).

The author gives all solutions x of the congruence $\tau(x) \equiv a \pmod{k}$, $k = 3, 4$. These results can be obtained at once from a paper of K. G. Ramanathan [cf. the preceding review]. Corresponding results for the modulus 8 are said to be too lengthy for inclusion in the present note.

D. H. Lehmer (Berkeley, Calif.).

Gupta, Hansraj. A congruence relation between $\tau(n)$ and $\sigma(n)$. J. Indian Math. Soc. (N.S.) **9**, 59–60 (1945).

The author proves that if $\tau(n)$ is Ramanujan's function defined by

$$x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24} = \sum_{n=1}^{\infty} \tau(n)x^n,$$

if $\sigma(n)$ is the sum of the divisors of n and if r is any divisor of 24 prime to n then $\tau(n)$ and $\sigma(n)$ are congruent modulo r. [Cf. H. Gupta, Proc. Benares Math. Soc. **5**, 17–22 (1943); K. G. Ramanathan, Proc. Indian Acad. Sci., Sect. A. **19**, 146–148 (1944); Math. Student **11**, 33–55 (1943); these Rev. **7**, 50; **6**, 37.] *D. H. Lehmer* (Berkeley, Calif.).

Banerjee, D. P. On the new congruence properties of the arithmetic function $T(n)$. Proc. Nat. Acad. Sci. India. Sect. A. **12**, 149–150 (1942).

For Ramanujan's function [cf. the preceding review] the author notes the congruence properties

$$\tau(2^n m) \equiv 0 \pmod{8^n}, \quad \tau(7^n m) \equiv 0 \pmod{7^n}, \quad \tau(10m) \equiv 0 \pmod{40}.$$

D. H. Lehmer (Berkeley, Calif.).

Jackson, F. H. Basic functions and polynomial sequences. Quart. J. Math., Oxford Ser. **17**, 99–110 (1946). [MF 16776]

The author introduces the following two sequences of polynomials: $f_n(x) = f_{n-1}(x) + xf_{n-2}(x)$, $f_0(x) = 0$, $f_1(x) = 1$, and $F_n(x) = c_1 f_{n-1}(x) + c_2 f_{n-2}(x)$. Several identities are proved and applications to the theory of numbers indicated.

P. Erdös (Syracuse, N. Y.).

Rado, R. The irreducible factors of certain polynomials. Quart. J. Math., Oxford Ser. **17**, 111–115 (1946). [MF 16777]

The author investigates the polynomials $f_n(x)$ introduced by Jackson [see the preceding review]. He proves that there are irreducible polynomials $H_m(x)$, $m \geq 3$, with integer coefficients such that $f_n = \prod_{m|n} H_m$, $m \geq 3$, $n \geq 3$. Also, if $m \neq m_1$, H_m and H_{m_1} are coprime. Some formulas are given for the coefficients of $H_m(x)$. *P. Erdös.*

Kloosterman, H. D. Partitions. Euclides, Groningen **21**, 67–77 (1946). (Dutch)

An expository lecture.

Trost, E. Ein wichtiger Begriff der additiven Zahlentheorie. Elemente der Math. **1**, 57–60 (1946).

This is an introduction to the elementary aspects of the $\alpha+\beta$ hypothesis [cf. Erdös and Niven, Amer. Math. Monthly **53**, 314–317 (1946); these Rev. **7**, 507].

P. Scherk (Saskatoon, Sask.).

Barbilian, D. Modernisierung des Beweises des Dirichlet-Jacobischen Satzes. Mathematica, Timişoara 22, 159–169 (1946).

The modernization begins with the use of the Wedderburn theorem that every finite skew field is commutative, instead of the more old fashioned result that $x^2+y^2+1\equiv 0 \pmod p$ is solvable for every prime p. From this the author proves that every integer is representable as a sum of four squares (a result first proved by Lagrange). He does not prove the theorems of Jacobi and Dirichlet on the number of representations but, using the arithmetic of integral quaternions, he derives related theorems on the enumeration of right ideals of odd square-free norm. *M. Hall.*

Giuga, G. Sopra alcune proprietà caratteristiche dei numeri primi. Period. Mat. (4) 23, 12–27 (1943).

By means of simple modifications and extensions of Wilson's theorem the author derives several alternative sets of necessary and sufficient conditions for a number p to be a prime. Other tests for primality involving binomial coefficients are also given. *R. A. Rankin.*

Storchi, E. Di un teorema sul determinante di Vandermonde dei primi n numeri della serie naturale. Period. Mat. (4) 22, 32–36 (1942).

Wilson's theorem is used to show that a necessary and sufficient condition for an odd number n to be a prime is that $D_n^2+(-1)^{\frac{1}{2}(n-1)}\equiv 0 \pmod n$, where D_n is Vandermonde's determinant $\|a_{pq}\|=\|p^q\|$, $1\leqq p\leqq n$, $0\leqq q\leqq n-1$. *R. A. Rankin* (Cambridge, England).

Buck, R. C. Prime-representing functions. Amer. Math. Monthly 53, 265 (1946). [MF 16451]

The following theorem is proved. If a rational function takes prime values at the integers, it must be constant. [The combination of two problems [sect. 8, problems 93, 97, pp. 133, 134] from Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis [Springer, Berlin, 1925] will lead immediately to the theorem.] *A. J. Kempner.*

Fueter, Rud. Über primitive Wurzeln von Primzahlen. Comment. Math. Helv. 18, 217–223 (1946).

Circa 1750 Euler [Opera Omnia, ser. I, vol. 5, Geneva, 1944, pp. xxi ff.; these Rev. 8, 2] discovered intuitively criteria for determining when 2, 3, 5 and 7 are cubic residues modulo p, a prime. These criteria, which are included in Eisenstein's general law of cubic reciprocity, are used by the author, together with Gauss's law of quadratic reciprocity, in order to find in what circumstances 2, 3, 5 and 7 are primitive roots modulo p, when p has one of the following three forms: $p=1+2^{2n}\cdot 3^{2m+1}$ (5, 7 primitive roots), $p=1+12q$ (2 a primitive root) and $p=1+6q\neq 13$ (3, 5 and 7 primitive roots). In the last two cases q is a prime. Previous results of this type by Chebyshev and Wertheim had used only the quadratic law of reciprocity.

R. A. Rankin (Cambridge, England).

Schmid, Hermann Ludwig, und Teichmüller, Oswald. Ein neuer Beweis für die Funktionalgleichung der L-Reihen. Abh. Math. Sem. Hansischen Univ. 15, 85–96 (1943). [MF 15830]

Let K be an algebraic function field of one variable with a finite coefficient field k with q elements. If F is an integral divisor of K, a character χ with conductor F is a homomorphic mapping of the group of divisors prime to F into a finite multiplicative group of roots of unity; the statement that F is the conductor means that $\chi(\xi)=1$ for every principal divisor (ξ) with $\xi\equiv 1 \pmod F$, but that this condition fails for any proper divisor of F. The L-function of χ is defined as $L(s,\chi)=\sum\chi(A)N(A)^{-s}$, where the sum is taken over all integral divisors A prime to F with absolute norms $N(A)$. The functional equation asserts that

$$L(1-s,\bar\chi)=\bar\chi(WF)W_0(\chi)q^{1-s-f}q^{(f+2g-2)s}L(s,\chi);$$

here W is a divisor of a suitable differential $d\omega$ of K, $W_0(\chi)$ is a complex number depending only on χ, g is the genus of K and f is the degree of the divisor F. This equation, which was conjectured by Hasse [J. Reine Angew. Math. 172, 37–54 (1934)], was proved by Weissinger [same Abh. 12, 115–126 (1937)]. The new proof proceeds by a reduction of the equation to an identity in the ring of all functions of K integral at the prime divisors of F, modulo the elements divisible by F. The treatment of this finite ring depends on "dual" moduli [cf. Teichmüller, J. Reine Angew. Math. 185, 1–11 (1943); these Rev. 5, 37]. *S. MacLane.*

Linnik, U. V. On the density of the zeros of L-series. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 35–46 (1946). (Russian. English summary)

The author proves the approximate functional equation for Dirichlet L-series: if $c>0$ is a suitable constant, $\chi(n)$ a primitive character mod q, $0\leqq\sigma\leqq 1$, $|t|>c$, $x>cq$, $y>c$, $2\pi xy=q|t|$, then

$$L(s,\chi)=\sum_{n<x}\chi(n)n^{-s}+\alpha(s,\chi)\sum_{n<y}\bar\chi(n)n^{s-1}+O(q^{\frac{1}{2}}(x^{-\sigma}+y^{\sigma-1}|t|^{\frac{1}{2}-\sigma})\log t),$$

where $\alpha(s,\chi)$ is defined by the functional equation $L(s,\chi)=\alpha(s,\chi)L(s-1,\bar\chi)$. The author proves this uniformly in σ, t, q and χ. For fixed values of q, the result is contained in a theorem by Suetuna [Jap. J. Math. 9, 111–116 (1932)].

Following a method applied to the ζ-function by Titchmarsh [Proc. London Math. Soc. (2) 30, 319–321 (1929)], the author then proves that for $0<\nu<\frac{1}{2}$, $T\geqq q^{50}$ there are at most $O(q^{2\nu}T^{1-\nu/(1-\nu)}\log^{10}T+q^{20})$ zeros of $L(s,\chi)$ in the rectangle $\frac{1}{2}+\nu\leqq\sigma\leqq 1$, $|t|\leqq T$; this holds uniformly in ν, T, q and χ. This result has already been applied by the author in a previous paper on Goldbach's problem [C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 3–7 (1945); these Rev. 7, 507]. *H. A. Heilbronn* (Bristol).

Tchudakov, N. Sur les zéros des L-fonctions de Dirichlet. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 89–91 (1945). [MF 16402]

The author states an extension of his previous result [Rec. Math. [Mat. Sbornik] N.S. 1(43), 591–602 (1936)] on the zeros of Dirichlet's L-functions. It is that there exist positive absolute constants C_1, γ ($\gamma<1$) such that $L(s,\chi)\neq 0$ for $0<|t|\leqq t_0$, $\sigma\geqq 1-C_1(\log k)^{-1}$; $|t|\geqq t_0$, $\sigma\geqq 1-C_1(\log|t|)^{-\gamma}$, where $(\log t_0)^\gamma=\log k$. From this result it follows, by classical methods, that, if $k<(\log x)^A$ for any fixed A, then

$$\pi(x;k,l)=(\mathrm{li}\,x)/\phi(k)+O\{x\exp(-C_2(\log x)^\mu)\},$$

where $\mu=(1+\gamma)^{-1}$. Finally, by generalising to the L-functions Ingham's estimate [Quart. J. Math., Oxford Ser. 8, 255–266 (1937)] for the number of zeros of the zeta function, the author proves that

$$\pi(x+x^\theta;k,l)-\pi(x;k,l)\sim x^\theta/(\phi(k)\log x)$$

for $k<(\log x)^A$, where $\theta=\frac{3}{4}+\epsilon$. *H. Davenport* (London).

Mahler, Kurt. Lattice points in two-dimensional star domains. I. Proc. London Math. Soc. (2) **49**, 128–157 (1946).

A star domain K is defined to be a closed bounded set, containing the origin O as an inner point, which is symmetric with respect to O and such that its boundary is cut in exactly two points by every straight line through O. A lattice is defined to be K-admissible if its only point interior to K is O. Let $\Delta(K)$ be the lower bound of the determinants of all K-admissible lattices. If the determinant of a lattice is $\Delta(K)$, the lattice is said to be critical. The boundary of K is assumed to consist of a finite number of analytic arcs.

The purpose of the paper is to determine a process whereby $\Delta(K)$ may be found for a given star domain K in a finite number of steps. The author shows easily that a critical lattice contains at least two independent boundary points P_1, P_2 of K and then that there is only a finite number of K-admissible lattices containing both P_1 and P_2. Critical lattices are defined to be singular or regular according as they contain exactly four or more than four boundary points of K. An inner tac line is a straight line through a boundary point P of K such that all its points within a neighborhood of P lie in K. If a lattice containing P_1, P_2 is singular then the line through P_2 parallel to OP_1 is shown to be an inner tac line. This leads to two equations connecting P_1 and P_2 which, in general, determine a finite number of possible positions for the two points because of the analytic character of the boundary. This in turn leads to the determination of every possible K-admissible lattice with these two points and so to all the singular lattices. A somewhat similar argument leads to the determination of all regular lattices. The relations used in this case are the minimal condition for the determinant of the lattices and the equations derived from the boundedness of the number of lattices with three or more given boundary points. Several special cases arise because of lattice points located at the end points of the component arcs of the boundary and because of the coincidence of certain analytic functions in the solution of the equation systems.

An example of a domain with a singular lattice is given but actual applications of the method are left for part II. The author points out that the lattice problem for nonsymmetric domains and for infinite star domains K containing a bounded star domain K' with a critical K-admissible lattice can easily be reduced to the problem for bounded star domains. *D. Derry* (Vancouver, B. C.).

Mahler, K. Lattice points in n-dimensional star bodies. II. Reducibility theorems. I, II. Nederl. Akad. Wetensch., Proc. **49**, 331–343, 444–454 = Indagationes Math. **8**, 200–212, 299–309 (1946). [MF 16827]

[Part I is in the course of publication in Proc. Roy. Soc. London. Ser. A.]

Let $F(x_1, \cdots, x_n) = F(X)$ be a continuous function over Euclidean n-space which assumes positive values except at the origin, where it vanishes, and has the property that $F(tX) = |t| F(X)$ for all real t. Let H be the star body of all points X for which $F(X) \leqq 1$. Let $\Delta(H)$ denote the lower bound of the determinants of all H-admissible lattices (that is, lattices whose only point interior to H is the origin). An H-admissible lattice with determinant $\Delta(H)$ is said to be critical. A star body H which contains a star body K as a proper subset such that $\Delta(H) = \Delta(K)$ is said to be reducible, otherwise irreducible.

If a star body is irreducible the author shows that a critical lattice goes through every boundary point. Sufficient conditions are given under which a bounded star body is irreducible. This second set of conditions is used to show that some simple convex bodies in 2 and 3-space are irreducible, among them the square, sphere and cube, and also to develop a construction whereby additional convex irreducible bodies in 2-space may be constructed. Some nonconvex irreducible star bodies are listed without proofs of irreducibility. The constant ω_{II} is defined to be the maximum of $F(X_1+X_2)$, where X_1 and X_2 are points in n-space for which $F(X_1)+F(X_2)=1$. Some inequalities involving the critical lattices of a star body and ω_{II} are developed. The author derives some results on a class of unbounded star bodies and uses these to give a simple construction for a star body which has critical lattices without points on its boundary. Unsolved problems are proposed on every topic discussed, among them the following. Does every bounded star body H contain an irreducible star body K with $\Delta(H) = \Delta(K)$? *D. Derry* (Vancouver, B. C.).

Mahler, K. Lattice points in n-dimensional star bodies. II. Reducibility theorems. III, IV. Nederl. Akad. Wetensch., Proc. **49**, 524–532, 622–631 = Indagationes Math. **8**, 343–351, 381–390 (1946).

This is the concluding part of the paper reviewed above. The author defines an infinite star body K to be boundedly reducible if it contains a bounded star body H with $\Delta(H) = \Delta(K)$. If no such body H exists, K is said to be boundedly irreducible. An example of a boundedly irreducible body of dimension n is given. A critical lattice Λ of K is defined to be strongly critical if a bounded star body K^* exists within K such that, for every K^*-admissible lattice Λ^* sufficiently close to Λ, $d(\Lambda^*) \geqq d(\Lambda)$. The author shows that, if every critical lattice of K is strongly critical, K is boundedly reducible. This criterion is applied to show that the domains $|x_1x_2x_3| \leqq 1$, $(x_1^2+x_2^2)|x_3| \leqq 1$ are each boundedly reducible. The proofs use the determination of the critical lattices of these domains given by H. Davenport [Proc. London Math. Soc. (2) **44**, 412–431 (1938); **45**, 98–125 (1939)]. As a consequence of the second of these results the author derives the following approximation theorem. If α_1, α_2 are irrational numbers, arbitrarily large integers u_1, u_2, u_3 exist so that

$$(u_1/u_3-\alpha_1)^2+(u_2/u_3-\alpha_2)^2 \leqq (2/\sqrt{23})\,|u_3|^{-3}.$$

The final result of the paper is that, if $F_r(X) \to F(X)$ as $r \to \infty$ uniformly for X on the unit hypersphere $|X| = 1$, then $\liminf \Delta(K_r) \geqq \Delta(K)$; where K_r is the star body of all points X for which $F_r(X) \leqq 1$. An example is given for which $\lim \Delta(K_r)$ exists and exceeds $\Delta(K)$. *D. Derry.*

THEORY OF GROUPS

Misra, Rama Dhar. Lattice sums of cubic crystals. Proc. Nat. Acad. Sci. India. Sect. A. **13**, 275–283 (1943).

Schon in früheren Arbeiten [Misra, Proc. Cambridge Philos. Soc. **36**, 173–182 (1940); Born und Misra, ebenda, 466–478 (1940); diese Rev. **1**, 277; **2**, 144] betrachtete der Verfasser Summen der Form

$$S_n^{(h_1,h_2,h_3)} = \sum_l{}' \frac{l_1^{2h_1} l_2^{2h_2} l_3^{2h_3}}{(l_1^2+l_2^2+l_3^2)^{h_n}},$$

erstreckt über alle Punkte eines kubischen Gitters mit Ausnahme des Ursprungs. In dieser Arbeit werden $S_n^{(0)}$ und $S_n^{(2)}$ abgeschätzt. Man zerlegt eine solche Summe in einen ersten Teil, der alle Punkte innerhalb einer bestimmten Entfernung vom Ursprung umfasst und nimmt dabei diese Entfernung so, dass dieser Teil numerisch ausgewertet werden kann. Der zweite Teil wird mittels Integralen nach oben und unten abgeschätzt, ferner werden Näherungswerte dafür angegeben. Mittels $S_n^{(0)}$ und $S_n^{(2)}$ können durch die Rekursionsformeln $3S_n^{(1)}=S_{n-2}^{(0)}$ und $3S_n^{(2)}+6S_n^{(1,1)}=S_{n-4}^{(0)}$ auch $S_n^{(1)}$ und $S_n^{(1,1)}$ abgeschätzt werden.

J. J. Burckhardt (Zürich).

Miller, G. A. Prime number of operators in sets of conjugates. Proc. Nat. Acad. Sci. U. S. A. **32**, 149–152 (1946). [MF 16710]

[Cf. the same Proc. **32**, 53–56 (1946); these Rev. **7**, 371.] The author further emphasises the fact that, if the number of noninvariant operators in any conjugate set of a group G is prime, then the order of G is divisible by at most two distinct primes. *G. de B. Robinson* (Toronto, Ont.).

***Piccard, Sophie. Sur les bases du groupe symétrique et les couples de substitutions qui engendrent un groupe régulier.** Mém. Univ. Neuchâtel, vol. 19. Libraire Vuibert, Paris, 1946. 223 pp.

The book is divided into two parts. In part I are gathered together results obtained by the author concerning the generation of the symmetric group $\mathfrak{S}_n$ and the alternating group $\mathfrak{A}_n$ by two substitutions S, T. A number of general theorems are deduced of which the following is typical. If $n(>9)$ is even and a, b, c, d, e are any five distinct integers not exceeding n, then a necessary and sufficient condition that $S=(1\,2\cdots n)$, $T=(abcde)$ should generate $\mathfrak{S}_n$ is that the highest common factor of ab, ac, ad, ae and n should be unity. A list of all possible choices for S, T, where $n\leqq 6$, is given at the end of part I.

Part II is concerned with regular groups and their generation by two regular substitutions S, T. To discuss this question the author introduces what she calls property p_r. Let the order of S be m and its degree km, where k and m are both greater than 1, and let T be a substitution of the same degree on the same symbols as S. Then T has property p_r with regard to S if T transforms the elements of each cycle of S into elements of r other cycles of S. For example, $T=(14)\ (27)\ (3\,10)\ (58)\ (6\,12)\ (9\,11)$ has property p_3 with regard to $S=(123)\ (456)\ (789)\ (10\,11\,12)$. The author shows that, for any $r\leqq 1$, there exist regular substitutions S, T, such that T has property p_r with regard to S and the group (S, T) is regular. For the case $r=1$, necessary and sufficient conditions are obtained for (S, T) to be regular. If $r\geqq 2$ the situation is more complicated and the results less general in character. *G. de B. Robinson.*

Lubelski, S. Verallgemeinerung eines Frobeniusschen gruppentheoretischen Satzes. Actas Acad. Ci. Lima **8**, 133–137 (1945).

Let G be a finite group of order H; let $p_1, \cdots, p_s$ be prime divisors of H; let $P=p_1\cdot\cdots\cdot p_s$; let Q and M be multiples of P such that Q divides M, M divides H, and M divides some power of P; let A be any invariant complex (sum of conjugate classes) in G; let R be the set of all a in G such that $a^t\varepsilon A$ for some t which is simultaneously a multiple of Q and a divisor of M; and let K be the number of elements in R. Then K is divisible by Q/P. *R. M. Thrall.*

Levi, F. W. Notes on group theory. VII. The idempotent residue classes and the mappings $\{m\}$. J. Indian Math. Soc. (N.S.) **9**, 37–42 (1945).

[For notes I–II, III and IV–VI see the same J. (N.S.) **8**, 1–9, 44–56, 78–91 (1944); these Rev. **6**, 40, 202; **7**, 113.] Denote by $E(G)$ the set of all the integers m such that $x\to x^m$, for x in the group G, is an endomorphism of G and denote by M the smallest positive integer in $E(G)$ such that G^M is Abelian. The author proves the existence of mutually relatively prime integers q_i such that $M=q_1\cdots q_t$, $2<q_i$, and such that $E(G)$ consists of the integers which are congruent to 0 or 1 modulo q_i for $i=1, \cdots, t$. If, in particular, $G^M=1$, then G is the direct product of groups $G(i)$ such that $G(i)^{q_i}=1$ and there exist finite groups G meeting these requirements for preassigned mutually relatively prime integers $q_1, \cdots, q_t$ such that $2<q_i$. [Cf. note I of this series.] *R. Baer* (Urbana, Ill.).

Tchounikhin, S. A. Sur la structure compositionnelle des sous-groupes de Siloff pour les groupes simples. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 419–420 (1946).

The following generalisation of an earlier theorem of the author [Rec. Math. [Mat. Sbornik] N.S. **5(47)**, 537–543 (1939); these Rev. **1**, 161] is stated. Let $\mathfrak{H}$ be a subgroup of index k in a group $\mathfrak{G}$ and let $\mathfrak{K}$ be a subgroup of $\mathfrak{H}$ for which $\mathfrak{H}/\mathfrak{K}$ is Abelian. Suppose that A is an element of $\mathfrak{H}$ not belonging to $\mathfrak{K}$ such that for every λ which divides the order of A the elements of $\mathfrak{H}$ conjugate in $\mathfrak{G}$ to A^λ belong to the coset $\mathfrak{K}A^\lambda$, and let σ be the order of A relative to $\mathfrak{K}$. Then if σ does not divide k, $\mathfrak{G}$ is not simple. This result is used to obtain generalisations of known inequalities involving the numerical invariants of the Sylow subgroups of simple groups. They are too complicated to be reproduced here. *S. A. Jennings* (Vancouver, B. C.).

Kaloujnine, Léo. La structure du p-groupe de Sylow du groupe symétrique du degré p^2. C. R. Acad. Sci. Paris **222**, 1424–1425 (1946).

In a previous report [same C. R. **221**, 222–224 (1945); these Rev. **7**, 239] the author showed that the Sylow subgroup P_m of the symmetric group of degree p^m may be written as a group of permutations of m-dimensional vectors with components in $GF(p)$. When $m=2$ these permutations take the form $\{(x, y)\to(x+a, y+a(x))\}$, where $x, y, a\varepsilon GF(p)$ and $a(x)$ is a polynomial in $GF(p)$ of degree at most $p-1$; denote this permutation by $[a, a(x)]$. In the present note it is shown that P_2 may be generated by $[1, 0]$ and $[0, 1-x^{p-1}]$. It follows that, if $z_0\subset\cdots\subset z_p=P_2$ is the upper central series of P_2, then, for $0\leqq m\leqq p-1$, z_m consists of all permutations $[0, a(x)]$ with $a(x)$ of degree at most $m-1$, and that the elements of P_2 of order p^2 are precisely those of the form $[a, a(x)]$ with $a\neq 0$ and $a(x)$ such that $a(0)+\cdots+a(p-1)\neq 0$. Properties of the non-Abelian subgroups are also stated; in particular, the number of such subgroups which are transitive on the vectors (x, y) is determined. *S. A. Jennings* (Vancouver, B. C.).

Dubuque, P. Sur les automorphismes des p-groupes. Rec. Math. [Mat. Sbornik] N.S. **18(60)**, 281–298 (1946). (Russian. French summary)

The author gives limitations on the order a of the group of automorphisms of a group $\mathfrak{P}$ of order p^n (p prime), on the assumption that $\mathfrak{P}$ has a minimal basis $P_1, P_2, \cdots, P_d$ (set of generators with minimal number of terms) possessing special properties. Let there be n_i of the $\mathfrak{P}$'s of order p^{m_i},

$i=1, 2, \cdots, r$; $m_1>m_2>\cdots>m_r$; $\sum n_i=d$. Put

$$\psi(k)=(p^k-1)(p^{k-1}-1)\cdots(p-1).$$

If the P's have certain properties B and C, then a divides $p^{kd(2n-d-1)}\psi(n_1)\psi(n_2)\cdots\psi(n_r)$; if the P's have properties A, C and D, then a divides $p^{kd(2n-d-1)}\psi(n_1)\psi(d-n_1)$. Under the same conditions, the order of an automorphism of $\mathfrak{P}$, if prime to p, must divide $\psi(n_k)$, where $n_k=\max(n_1\cdots n_r)$. The properties A–D are rather too complicated to state here. *P. Hall* (Cambridge, England).

Brauer, Richard. On blocks of characters of groups of finite order. I. Proc. Nat. Acad. Sci. U. S. A. **32**, 182–186 (1946). [MF 16859]

This is a continuation of a paper in the same Proc. **30**, 109–114 (1944) [these Rev. **6**, 34]. Let H be a subgroup of G of order p^k, where p is a fixed prime. Denote by $\mathfrak{C}(H)$ the centralizer of H in G and by $\mathfrak{N}(H)$ the normalizer of H in G, and let N be a subgroup such that $H\mathfrak{C}(H)\subseteq N\subseteq\mathfrak{N}(H)$. If the block $\bar{B}_\sigma$ of N with the defect group $\bar{H}_\sigma$ determines the block B_τ of G with the defect group H_τ, then $H\subseteq\bar{H}_\sigma\subseteq N$ and $\bar{H}_\sigma$ is conjugate in G to a subgroup of H_τ. Let $(\mathfrak{H})$ be a system of subgroups of H of orders $1, p, p^2, \cdots$ of G such that every subgroup of order p^k of G is conjugate to exactly one H in $(\mathfrak{H})$. A rule is established by which a block B of G is determined by a block $\bar{B}$ of N and the concept of the classes of G associated with a given block is introduced. For every H in $(\mathfrak{H})$, one finds the collection $\bar{B}^{(\tau)}$ of all blocks $\bar{B}_\sigma$ of $\mathfrak{N}(H)$ which determine a given block B_τ of G and selects a full system of classes $\bar{K}_\rho$ of $\mathfrak{N}(H)$ which are associated with $\bar{B}^{(\tau)}$. Suppose that $r_\tau(H)$ of these classes $\bar{K}_\rho$ have H as their defect group. Different ones of these $r_\tau(H)$ classes $\bar{K}_\rho$ belong to different classes K_α of G; the classes K_α thus obtained for the different H in $(\mathfrak{H})$ form a possible selection of classes associated with B_τ. The number of characters in B_τ is $x_\tau=\sum r_\tau(H)$, the sum extending over all H in $(\mathfrak{H})$. If $s_\tau(H)$ of the $r_\tau(H)$ classes $\bar{K}_\rho$ are p-regular, the number of modular characters in B_τ is similarly $y_\tau=\sum s_\tau(H)$. If the sum extends only over those groups of $(\mathfrak{H})$ which have a fixed order p^k, the corresponding sum represents the multiplicity of p^k as an elementary divisor of the Cartan matrix C_τ of the block B_τ. The numbers $r_\tau(H)$ and $s_\tau(H)$ depend only on the group G, the subgroup H and the block B_τ. The classes of defect 0 in G form the basis of a subalgebra M of the center Δ^* of the modular group ring Γ^* of G. The number of blocks of defect 0 is equal to the rank of M^* for sufficiently large n. *C. C. MacDuffee.*

Baer, Reinhold. Absolute retracts in group theory. Bull. Amer. Math. Soc. **52**, 501–506 (1946). [MF 16810]

The subgroup R of the group G is a retract of G when an idempotent endomorphism of G exists which maps G on R. This definition is analogous to the concept of retract in topology. The following theorems are proved. A group G is a retract of every group containing G as a normal subgroup, if and only if G is complete (that is, G has only inner automorphisms and the center of G consists of the identity). If G is a retract of every group containing G, then G consists of the identity only. Every subgroup of the group G is a retract of G, if and only if G is an Abelian group whose elements have finite orders which are not divisible by the squares of prime numbers. If N is a normal subgroup of a group G, the group G/N is said to be retractable when there exists an idempotent endomorphism of G with the kernel N. If every quotient group of G is retractable, then the group G is called a totally retractable group. Some properties of this class of groups are obtained. *R. Brauer.*

Loonstra, F. Ordered groups. Nederl. Akad. Wetensch., Proc. **49**, 41–46 = Indagationes Math. **8**, 5–10 (1946). [MF 16561]

The author proves the known result that every Archimedean ordered group is order-isomorphic with an additive group of real numbers [cf. O. Hölder, Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. **53**, 1–64 (1901), in particular, pp. 13–14; R. Baer, J. Reine Angew. Math. **160**, 208–226 (1929), in particular, p. 217; H. Cartan, Bull. Sci. Math. **63**, 201–205 (1939); these Rev. **1**, 43].

G. Birkhoff (Cambridge, Mass.).

Godement, Roger. Extension à un groupe abélien quelconque des théorèmes taubériens de N. Wiener et d'un théorème de A. Beurling. C. R. Acad. Sci. Paris **223**, 16–18 (1946).

Two theorems are stated and the proofs are sketched. The first asserts that a closed ideal in the group algebra A of a locally compact Abelian group (that is, the algebra of complex-valued functions on the group which are integrable relative to Haar measure, multiplication being defined as convolution) coincides with A provided it has the property that for each point in the (topological) character group there is an element of the ideal whose Fourier transform is not zero at the point. In the special case that the group is the additive group of the reals, the theorem is essentially identical with one of the two theorems known collectively as the general Tauberian theorem of Wiener [The Fourier Integral and Certain of its Applications, Cambridge University Press, 1933, p. 37, theorem 4]. The second theorem, which is indicated to follow readily from the first, has as a corresponding special case a theorem of A. Beurling [Acta Math. **77**, 127–136 (1945); these Rev. **7**, 61]. The first theorem is a fairly easy consequence of a theorem of the reviewer [Proc. Nat. Acad. Sci. U. S. A. **27**, 348–352 (1941), theorem 2; these Rev. **3**, 36]. *I. E. Segal.*

Følner, Erling. W-almost-periodic functions on arbitrary groups. Mat. Tidsskr. B. **1946**, 153–162 (1946). (Danish)

In analogy to the classical almost periodic functions on the axis of real numbers in the sense of Weyl [Math. Ann. **97**, 338–356 (1926)] the author defines the class W^p_{ap} of almost periodic functions on a group in the following way: $f\in W^p_{ap}$ if and only if there exists a sequence of "trigonometric polynomials on the group" $\{S_n\}$ such that

$$\lim_{n\to\infty}\inf_{(\alpha_\tau,\sigma_\tau)}\sup_{(x,y)}\left\{\sum_\tau\alpha_\tau|f(x\alpha_\tau y)-s_n(x\sigma_\tau y)|^p\right\}^{1/p}=0.$$

He shows that this class is identical with the following class W^p_{lap}: $f\in W^p_{lap}$ if and only if to every $\epsilon>0$ there exist a finite number of elements m_ϱ, a_τ, and numbers $\alpha_\tau>0$ such that $\sum\alpha_\tau=1$ and to every element t there exists a $q(t)$ such that

$$\sup_{(x,y)}\{\sum_\tau\alpha_\tau|f(xta_\tau y)-f(xm_{q(t)}a_\tau y)|^p\}^{1/p}<\epsilon.$$

František Wolf (Berkeley, Calif.).

Borůvka, O. Grupoidentheorie. I. Publ. Fac. Sci. Univ. Masaryk **1939**, no. 275, 17 pp. (1939). (Czech. German summary)

Unter einem Grupoid verstehe ich den Inbegriff einer nicht leeren Menge G und einer Multiplikation in G. Die

Kenntnis der Grupoidentheorie gestattet begreiflich einen tieferen Einblick in die Struktur der gruppentheoretischen Sätze und lässt insbesondere den Einfluss der die Eigenschaften der Multiplikation beschreibenden Gruppenaxiome auf dieselben erkennen. In diesem ersten Teile handelt es sich um vorbereitende Betrachtungen über Zerlegungen von Mengen und um die wichtigsten allgemeinen Begriffe und Sätze über Grupoide. Insbesondere wird hier eine Theorie der homomorphen Abbildungen von Grupoiden dargestellt.

From the author's summary.

Dubreil, Paul. Contribution à la théorie des demi-groupes. Mém. Acad. Sci. Inst. France (2) **63**, no. 3, 52 pp. (1941).

A demigroup is a system closed under an associative binary operation: $a(bc)=(ab)c$. It is called a semigroup if we can "simplify" on both sides: $ab=ac$ or $ba=ca$ implies $b=c$. An equivalence relation R in a demigroup D is called right regular if $a\equiv b \pmod R$ implies $ax\equiv bx \pmod R$ for any x in D; R is called right simplifiable if $ac\equiv bc\ (R)$ implies $a\equiv b\ (R)$. The terms "left regular" and "left simplifiable" are defined similarly; R is regular (simplifiable) if it is regular (simplifiable) on both sides. If R is regular, then the classes of elements of D equivalent modulo R can be multiplied together in the usual fashion and constitute a demigroup D/R homomorphic with D. If R is simplifiable, D/R is a semigroup.

In a group G any right regular equivalence relation R is right simplifiable, and vice versa, and coincides with the right coset decomposition modulo a subgroup E of G. If R is regular, E is invariant. The state of affairs in a demigroup is, however, not so simple. The author attacks the general problem by making a thorough study of two types of equivalence, which he calls principal and reversible.

Let H be any complex (that is, any subset) of the demigroup D and let a be an element of D. The right quotient of H by a is defined to be the set Q_a of all x in D such that ax is in H. The principal right equivalence R_H induced by H is defined as follows: $a\equiv b\ (R_H)$ if and only if $Q_a=Q_b$; R_H is right regular. The right residue W_H of H is the set of all w in D such that $Q_w=0$ (the empty set); W_H is a class mod R_H. Definitions of left quotient ${}_aQ$, principal left equivalence ${}_HR$ and left residue ${}_HW$ are similar.

Assume now that H is strong, that is, if aa', ab' and ba' are in H then bb' must be in H. (In a group such a complex must be a coset of some subgroup.) If $ac\equiv bc\ (R_H)$ and if ac and bc are not in W_H, then $a\equiv b\ (R_H)$. Using this, the author develops several results concerning right simplification mod R_H from properties of W_H. Conversely, let R be any right regular equivalence relation in D, and let H be any class mod R. Then $R\subseteq R_H$, that is, $a\equiv b\ (R)$ implies $a\equiv b\ (R_H)$. If, moreover, R is right simplifiable, then H is strong and $R=R_H$ outside of W_H.

The author then gives a few "homomorphism theorems" of which we state one. A subdemigroup S of D is called normal if (1) S is strong, (2) $R_S={}_SR$ and (3) $W_S={}_SW=0$. If S is normal then D/R_S is a group. Suppose conversely that D is homomorphic to a group G, and let R denote the corresponding equivalence relation in D. Then the class U of elements of D mapping into the identity element of G is a normal subdemigroup of D and $R=R_U={}_UR$. (For $G=D/R_S$, $S\subseteq U$.)

While a principal equivalence is induced by any complex in D, a reversible equivalence is defined only for a subdemigroup S of D satisfying the following condition: S is called right reversible if to each pair of elements a, b of S there exist elements x, y in S such that $xa=yb$. The right reversible equivalence P_S induced by S is defined as follows: $a\equiv b\ (P_S)$ if and only if s and s' exist in S such that $sa=s'b$; P_S is right regular. For several different combinations of conditions on S the author finds $P_S\subseteq R_S$, that is, under quite general circumstances P_S is a refinement of R_S.

The author concludes with a theory of inner automorphisms in the case that D is a semigroup. The interior I of D is defined to be the set of all elements a of D such that the equation $ax=ya$ is solvable in x for given y and in y for given x. The resulting one-to-one correspondence $x\rightarrow y$ of D with itself is called the inner automorphism induced by a. If not vacuous, I is a subsemigroup of D containing the center Z of D. The inner automorphisms of D form a semigroup J homomorphic with I. Let Σ denote the corresponding equivalence relation in I and let $\bar{P}_Z$ and $\bar{R}_Z$ denote the reversible and principal equivalences defined by Z in I. Then $\bar{P}_Z\subseteq\Sigma\subseteq\bar{R}_Z$ with equalities holding outside of the residue $\bar{W}_Z$ of Z in I. If $\bar{W}_Z=0$ then J is a group isomorphic with $I/\bar{R}_Z$. This generalizes a well-known theorem for groups G, namely that $J\cong G/Z$. *A. H. Clifford.*

ANALYSIS

Calculus

*de la Vallée Poussin, Ch. J. **Cours d'Analyse Infinitésimale.** Dover Publications, New York, N. Y., 1946. Vol. I, x+460 pp.; vol. II, x+524 pp. $8.00.

Photographic reproduction of the Louvain edition of 1938 [vol. I, 8th ed.; vol. II, 7th ed.].

Saltzer, Charles. A remark on the rectification of the Joukowski profile. Quart. Appl. Math. **4**, 196–199 (1946). [MF 16966]

The arc length of the Joukowski airfoil profile, which can be defined as the inversion of a parabola, is expressed in an integral formula that can be evaluated by elementary means. For the special case of the symmetrical profile, a simple formula is obtained and is integrated. *W. R. Sears.*

Pompeiu, D. Sur une proposition analogue au théorème des accroissements finis. Mathematica, Timişoara **22**, 143–146 (1946).

The usual geometric interpretation of the law of the mean of the differential calculus is that, if $f(x)$ is a function with a finite derivative everywhere in the closed interval $[a, b]$, then at some point interior to the interval the tangent line to the curve $y=f(x)$ will have the same slope as the chord connecting the points $(a, f(a))$ and $(b, f(b))$. The present paper shows that the term "slope" may be replaced by the term "y-intercept." *P. Civin* (Eugene, Ore.).

Cioranescu, Nicolas. Quelques formules d'équivalence ou de moyenne. Mathematica, Timişoara **22**, 91–101 (1946).

The following result is typical. Let $f_k(x)$, $k=1, 2, 3$, be integrable in (a, b) and let

$$\int_a^b f_1(x)dx < \int_a^b f_2(x)dx < \int_a^b f_3(x)dx.$$

Then there exists ξ, $a<\xi<b$, so that

$$\int_a^b f_3(x)dx=\int_a^t f_1(x)dx+\int_t^b f_2(x)dx.$$

R. P. Boas, Jr. (Providence, R. I.).

Nicolesco, Miron. Sur la seconde formule de la moyenne. Mathematica, Timişoara 22, 182–203 (1946).

In the first part the author derives the Bonnet form of the second mean value theorem for Lebesgue integrals from the Weierstrass form, which is well known to be valid for summable functions. [This result is not new; cf. E. W. Hobson, The Theory of Functions of a Real Variable, v. 1, 3d ed., Cambridge University Press, 1927, pp. 618–622.] Then the author generalizes the second mean value theorem for vectorial functions. In the second part he gives an extension of the second mean value theorem (in the Weierstrass form) to Lebesgue double integrals, after having established a generalization of integration by parts for double integrals. *A. Rosenthal* (Albuquerque, N. M.).

Levi, Beppo. On a formula of Laplace. Publ. Inst. Mat. Univ. Nac. Litoral 6, 341–351 (1946). (Spanish) [MF 16936]

The formula in question gives an asymptotic value for the integral

$$I_n=\int_a^b f(x)\{\varphi(x)\}^n dx,$$

when $a<x_0<b$, $f(x_0)\neq 0$ and $\varphi(x)$ has a positive absolute maximum value at x_0. The author assumes for convenience that $x_0=0$. The exact assumptions on $f(x)$ and $\varphi(x)$ are as follows: (1) $f(x)$ and $\varphi(x)$ are integrable; (2) $f(x)$ is bounded; (3) $f(x)$ is continuous at $x=0$ and $f(0)\neq 0$; (4) if $h>0$, there is a positive δ such that $|\varphi(x)|\leqq\varphi(0)-\delta$ when $|x|\geqq h$; (5) $\varphi(x)$ can be expressed in the form

$$\varphi(x)=\varphi(0)-c|x|^\nu\{1+\theta(x)\},$$

where c and ν are constants with $c>0$, $\nu>1$, and $\theta(x)$ is a function such that $\theta(x)\to 0$ as $x\to 0$. The conclusion, reached by entirely elementary reasoning, is that as $n\to\infty$

$$(*)\qquad I_n\sim 2\nu^{-1}\Gamma(1/\nu)\{\varphi(0)/c\}^{1/\nu}f(0)\{\varphi(0)\}^n/n^{1/\nu}.$$

This formula generalizes a result of Laplace and Darboux, to which it reduces on setting $\nu=2$, $c=-\frac{1}{2}\varphi''(0)$. It also generalizes a result of Stieltjes, corresponding to assumption (5) with the further assumptions $\nu=2$, $\theta(x)>0$. Under the additional assumptions that, as $x\to 0$, $\theta(x)=O(x^\rho)$ and $f(x)=f(0)+O(x^\rho)$, where $\rho>0$, the author gives the more precise result $I_n=A_n\{1+O(n^{-\rho/\nu+\epsilon})\}$, where ϵ is a positive arbitrarily small number and A_n is the asymptotic expression for I_n in (*). *A. E. Taylor* (Los Angeles, Calif.).

Roy, S. N. On a certain class of multiple integrals. Bull. Calcutta Math. Soc. 37, 69–76 (1945). [MF 15681]

The integrals are p-fold, with the integrand a determinant with typical term $x_i^{m_j}(1+x_i)^{-n_j}$. They were met in a statistical problem. A reduction formula is given which eventually leads to incomplete beta functions. *P. Franklin.*

Humbert, Pierre. Quelques séquences symboliques. Ann. Univ. Lyon. Sect. A. (3) 4, 85–92 (1941).

The author evaluates some definite integrals by means of the Laplace transform. *H. Pollard* (Ithaca, N. Y.).

Challier, J. Extension de la formule d'Ostrogradsky aux integrales non lineaires. Ann. Univ. Lyon. Sect. A. (3) 7, 46–56 (1944).

Let $f(x, y, z; X, Y, Z)=\xi(x, y, z)X+\pi Y+\zeta Z$ be a linear form in the bidifferentials $X=[dy, dz]$, $Y=[dz, dx]$, $Z=[dx, dy]$ and let R be a region bounded by the surface S. By a classical formula

$$(*)\qquad \iint_S f=\iiint_R\left(\frac{\partial\xi}{\partial x}+\frac{\partial\pi}{\partial y}+\frac{\partial\zeta}{\partial z}\right)[dx, dy, dz].$$

Instead of assuming that f is linear in X, Y, Z, the author assumes that f is homogeneous and of first degree in the bidifferentials. Interpreting $X/Y=-\partial y/\partial x$, etc., he gives a formula similar to (*) for this type of f. A similar result is given for n variables. *F. G. Dressel* (Durham, N. C.).

Theory of Sets, Theory of Functions of Real Variables

Piccard, Sophie. Note sur l'identité de deux ensembles de la famille F d'ensembles parfaits. Comment. Math. Helv. 18, 204–216 (1946).

Let k and n be any two integers such that $1\leqq k<n-1$; let $a_0, \cdots, a_k$ be any sequence of $k+1$ integers such that $a_0=0<a_1<\cdots<a_k$; let $[a_0, \cdots, a_k]_n$ be the set A of real numbers of the closed interval $(0, a_k/(n-1))$ which are representable in the n-ary scale of notation by the exclusive use of the digits $a_0, \cdots, a_k$. The set A is perfect and may be structurally defined by means of prescribed complementary intervals of successive stages $1, 2, 3, \cdots$. Let F be the family of sets constituted by A, varying as its definition permits. The purpose of this paper is to establish a criterion for the identity of two sets A: $[a_0, \cdots, a_k]_n$ and A': $[a'_0, \cdots, a'_\nu]_{n'}$ of F. This criterion generalizes a result of Henri Cartan and Hélène Cartan. The argument is based on the construction for A. *H. Blumberg.*

Keldych, L. Sur les transformations ouvertes des ensembles A. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 622–624 (1945).

Attention is confined to subsets of the Baire space (the set of irrational points of the interval (0, 1)). A continuous transformation of a set E into a set E_1 is said to be interior if it transforms sets open in E into sets open in E_1. Hausdorff [Fund. Math. 23, 279–291 (1934)] showed that the class of G_δ sets is closed under interior transformations. The results announced here show that this result is not true for higher classes in the Borel classification and also deal with interior transformations of A-sets into one another. These transformations are defined by setting up a correspondence between the Baire intervals from which the given sets are obtained by A-operations. For instance, it is stated that every A-set can be obtained by an interior transformation from certain sets which are both G_δ and F_σ or indeed from any set of this (or higher) type which is of the first category in itself. This last condition is essential, for the class of sets of the second category in themselves is closed under interior transformations. The problem as to whether all A-sets of the second category can be obtained from a particular one of this type is next considered. That the answer is negative in the case of an A-set whose complement has no perfect subset follows from the result of Hausdorff quoted above

and work of Mazurkiewicz [Fund. Math. 19, 198–204 (1932)]; the problem of the existence of such a set is, however, open. On the other hand, a class of A-sets is defined for each of which the answer is in the affirmative.

J. Todd (London).

Saarnio, Uuno. Uncountable well-ordering. Ajatus 13, 236–261 (1945). (Finnish)

The author claims to have constructed an uncountable well-ordered set of real numbers. The principle of construction is demonstrated for the simplest transfinite ordinals. There is no attempt to base the construction on transfinite induction and in order to do so it would be necessary to appeal to the axiom of choice. *L. Ahlfors.*

Sierpiński, Wacław. Sur une suite transfinie d'ensembles de nombres naturels. Fund. Math. 33, 9–11 (1945). [MF 16879]

Let N_α denote, for any ordinal α, an infinite set of natural numbers. The set N_2 is said to be almost included in N_1 if the set N_2-N_1 is finite or empty; N_1 and N_2 are said to be essentially different if the set $(N_1-N_2)+(N_2-N_1)$ is infinite. An inductive construction is given for a transfinite sequence $\{N_\xi\}$ of order-type Ω such that, for all $\alpha<\beta<\Omega$, the set N_β is almost contained in N_α and is also essentially different from it. *J. Todd* (London).

Sierpiński, Wacław. Un théorème sur les familles d'ensembles et ses applications. Fund. Math. 33, 1–6 (1945). [MF 16878]

Let $\mathfrak{m}$ be a fixed infinite cardinal. A family of sets will be said to have the property P if the product of any finite number of the sets has a cardinal not less than $\mathfrak{m}$. Let F be a family having this property and consisting of all the sets $E_1, E_2, \cdots, E_\omega, E_{\omega+1}, \cdots, E_\xi, \cdots$ $(\xi<\phi)$, where $\phi\geqq\omega$. Suppose that any decompositions $E_\xi=E_\xi^0+E_\xi^1$ of the sets E_ξ are given. Then an inductive construction is given of a sequence $\{\tau(\xi)\}$ $(\xi<\phi)$, where each $\tau(\xi)$ is 0 or 1, which is such that the family $\{E_\xi^{\tau(\xi)}\}$ $(\xi<\phi)$ has the property P. From this result, with an essential appeal to the axiom of choice, it follows that, given any infinite set H, there exists a function $f(E)$ which associates the value 0 or 1 with each subset E of H and which has the following properties: $f(E)$ is not identically zero, $f(E)$ is additive in the restricted sense (that is, $f(M+N)=f(M)+f(N)$ when M and N are disjoint subsets of H) and $f(E)=0$ if the cardinal of E is less than that of H.

Another consequence is the existence of a function of accumulation of order $\mathfrak{m}$ [see A. Tychonoff, Math. Ann. 111, 762–766 (1935)] for any family F of cardinal $\mathfrak{m}\geqq\aleph_0$ of real functions defined on an arbitrary set E and uniformly bounded thereon: this means that there exists a real function $f_0(x)$ such that for any $\epsilon>0$ and for any finite set of elements $x_i\in E$, $i=1, \cdots, n$, there are $\mathfrak{m}$ distinct functions $f(x)$ in F such that, for $i=1, \cdots, n$, $|f(x_i)-f_0(x_i)|<\epsilon$.

J. Todd (London).

Mostowski, Andrzej. Remarques sur la note de M. Sierpiński "Un théorème sur les familles d'ensembles et ses applications." Fund. Math. 33, 7–8 (1945).

A family of subsets of a set E is an ideal if it is hereditary and finitely additive; an ideal is prime if it contains either X or $E-X$ for any $X\subset E$. The first two results established by Sierpiński in the paper reviewed above remain true when we replace, in the definition of property P, "has a cardinal not less than $\mathfrak{m}$" by "does not belong to a given ideal." With this change the class of sets X such that $f(X)=0$ is a prime ideal containing the given one. Thus a new proof of what Tarski has called the fundamental theorem in the arithmetic of ideals is obtained.

The existence of a function of accumulation of order $\mathfrak{m}$ is shown to follow also from the result of Čech [Ann. of Math. (2) 38, 823–844 (1937), in particular, p. 830] that, if T is a bicompact topological space and E an arbitrary set, then E^T (the space of functions on E to T, topologized by defining as neighbourhoods of $f\in E^T$ the sets

$$\prod_{i=1}^{m} \mathrm{E}[g(x_i)\in G_i],$$

where, for $i=1, \cdots, m$, $x_i\in E$, and where G_i is open in T and $f(x_i)\in G_i$ is also bicompact. *J. Todd* (London).

Sierpinski, Waclaw. Exemple effectif d'une famille de $2^{\aleph_1}$ ensembles linéaires croissants. Rend. Accad. Sci. Fis. Mat. Napoli (4) 10, 2 pp. (1940). [MF 16872]

The object of this note is to define effectively a family F of $2^{\aleph_1}$ linear increasing sets (that is, any two of the sets of F are nonidentical and one set of the pair contains the other one). The construction proceeds from the definition of a function $f(D)$ mating with every denumerable set D of ordinals greater than or equal to ω and less than Ω a linear set $f(D)\neq 0$ such that $f(D)\cdot f(D')=0$ for $D\neq D'$. This construction derives from a modification of an idea used by H. Lebesgue [J. Math. Pures Appl. (6) 1, 139–216 (1905), in particular, p. 213] for effectively defining a sequence, of type Ω, of linear disjoint sets $(\neq 0)$. Corollary under the continuum hypothesis: the set of all real numbers is a sum of 2^c increasing sets, $c=2^{\aleph_0}$. *H. Blumberg.*

Sierpinski, Waclaw. Sur le paradoxe de MM. Banach et Tarski. Fund. Math. 33, 229–234 (1945). [MF 16875]

Proof of the following property of a (3-dimensional) sphere S (interior and boundary): S is decomposable into eight parts, without common points, of which five and three parts may be respectively decomposed into two spheres, without common points, of the same size as S, by suitable rigid displacements. Banach and Tarski [Fund. Math. 6, 244–277 (1924)] proved a like result for a finite number of parts without specifying their number; von Neumann [Fund. Math. 13, 73–116 (1929), in particular, p. 77] announced such a result with the numbers 8, 5 and 3 of the present paper replaced, respectively, by 9, 5 and 4. This type of result goes back to Hausdorff [Grundzüge der Mengenlehre, Leipzig, 1914, p. 469]. *H. Blumberg* (Columbus, Ohio).

Sierpinski, Waclaw. Sur la non-existence des décompositions paradoxales d'ensembles linéaires. Actas Acad. Ci. Lima 9, 113–117 (1946).

Deux ensembles de points A et B sont dits "congruents par décomposition finie $(A=_f B)$," s'il existe un nombre naturel n et des ensembles $A_1, \cdots, A_n$ et $B_1, \cdots, B_n$, tels que (1) $A=A_1+\cdots+A_n$, $B=B_1+\cdots+B_n$, (2) $A_kA_l=B_kB_l=0$ pour $1\leqq k<l\leqq n$, (3) $A_k\cong B_k$ pour $k=1, \cdots, n$, $A_k\cong B_k$ désignant que les ensembles A_k et B_k sont congruents (c'est-à-dire superposables par translation ou rotation). L'auteur donne une démonstration directe et élémentaire d'un théorème de A. Tarski [Fund. Math. 30, 218–234 (1938), p. 222, coroll. 1.17; 31, 47–66 (1938), p. 63, coroll. 3.14], c'est-à-dire: si E est un ensemble linéaire (non vide), il n'est pas possible que $E=C+D$, avec $E=_f C$, $E=_f D$, $CD=0$. *T. Viola* (Rome).

Sierpiński, Waclaw. Sur une suite infinie de fonctions de classe 1 dont toute fonction d'accumulation est non mesurable. (Solution d'un problème de M. S. Banach). Fund. Math. **33**, 104–105 (1945).

According to A. Tychonoff [Math. Ann. **111**, 762–766 (1935)], $f(x)$ is called a function of accumulation of the sequence $f_k(x)$, $k=1, 2, \cdots$, if for every $\epsilon>0$ and for every finite set $x_1, \cdots, x_m$ of real numbers there exist infinitely many values of k such that $|f(x_i)-f_k(x_i)|<\epsilon$ for $i=1, \cdots, m$. As Tychonoff proved, every uniformly bounded infinite sequence of real functions has at least one function of accumulation. Banach proposed the problem of whether every uniformly bounded infinite sequence of Baire functions has a measurable function of accumulation. The author answers this question in the negative by means of the sequence $f_k(x)=[2^k x]-2[2^{k-1}x]$, $k=1, 2, \cdots$, where $[a]$ designates the greatest integer not exceeding a.

A. Rosenthal (Albuquerque, N. M.).

Sierpiński, Waclaw. Sur les fonctions de plusieurs variables. Fund. Math. **33**, 169–173 (1945). [MF 16881]

Let F_n be the set of all functions $f(x_1, \cdots, x_n)$ of n variables x_i each of which ranges through any fixed set E which also contains the values of the functions. It is shown that any function of F_n (for $n=2, 3, \cdots$) can be obtained by superposing a finite number of functions of F_2. Two proofs are given for this in the case when E is finite. Each of these is more complicated than the proof given for the case when E is infinite. The last proof, in general, uses the axiom of choice. An appeal to this axiom can be avoided in the case when E is the set of all real numbers. [See W. Sierpiński, Prace Mat.-Fiz. **41**, 171–175 (1934); L. Bieberbach, J. Reine Angew. Math. **165**, 89–92 (1931); Hilbert's thirteenth problem, Gesammelte Abhandlungen, Berlin, 1935, p. 313.]

J. Todd (London).

Monna, A. F. Sur le problème de la mesure. Nederl. Akad. Wetensch., Proc. **49**, 63–64 = Indagationes Math. **8**, 27–28 (1946). [MF 16563]

Soit S un système de sous-ensembles d'un ensemble infini E tel que la réunion et l'intersection de tout nombre fini d'ensembles de S appartiennent à S et que E n'appartienne pas à S. Alors il est possible de faire correspondre à tout ensemble $V \subset E$ un nombre réel $m(V)$, appelé mesure, tel que (1) $m(V)$ est additive au sens restreint, (2) $m(E)=1$, (3) $m(V)=0$ pour tout $V \in S$, (4) $-1 \leqq m(V) \leqq +1$. Cas particulier: S est le système des ensembles $V \subset E$ dont le nombre cardinal est inférieur au nombre cardinal de E. Dans ce cas particulier le résultat précédent subsiste aussi en remplaçant (4) par $0 \leqq m(V) \leqq 1$.

A. Appert.

Borel, Émile. L'axiome du choix et la mesure des ensembles. C. R. Acad. Sci. Paris **222**, 309–310 (1946). [MF 16004]

The author constructs an example, similar to the simple classical example of Vitali [Sul Problema della Misura dei Gruppi di Punti di una Retta, Gamberini e Parmeggiani, Bologna, 1905], of the division of the unit circumference into a countable number of mutually disjoint congruent sets.

L. H. Loomis (Cambridge, Mass.).

Borel, Émile. Les ensembles homogènes. C. R. Acad. Sci. Paris **222**, 617–618 (1946). [MF 16038]

A set of real numbers is called homogeneous if it contains the sum and difference of any two of its members. The author constructs two noncountable (and nontrivial) homogeneous sets.

L. H. Loomis (Cambridge, Mass.).

Besicovitch, A. S., and Moran, P. A. P. The measure of product and cylinder sets. J. London Math. Soc. **20**, 110–120 (1945). [MF 16703]

The first part of the paper deals with the Hausdorff dimension of certain product sets. Let H^α be α-dimensional Hausdorff measure [Hausdorff, Math. Ann. **79**, 157–179 (1918)]. Suppose that X and Y are subsets of the line for which $0<H^\alpha(X)<\infty$, $0<H^\beta(Y)<\infty$. The following three theorems are proved. (1) $H^{\alpha+\beta}(X \times Y) \geqq H^\alpha(X)H^\beta(Y)$. (2) If Y has positive β-dimensional lower density at each of its points, then the Hausdorff dimension of $(X \times Y)$ is $\alpha+\beta$. (3) The Hausdorff dimension of $(X \times Y)$ never exceeds $\alpha+1$. The authors describe an example to show that this inequality is the best possible.

The second part of the paper concerns cylindrical sets. Let A be a subset of the plane, h a positive number and B the right cylinder with base A and height h. The authors indicate the construction of a set A for which $H^2(B)>hH^1(A)$. [This is somewhat surprising in view of the following known results. Randolph proved that $C^2(B) \leqq hC^1(A)=hH^1(A)$, where C^α is α-dimensional Carathéodory measure [Carathéodory, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1914**, 404–426; J. F. Randolph, Bull. Amer. Math. Soc. **42**, 268–274 (1936)]. Morse and Randolph proved that $G^2(B)=hG^1(A)$, where G^α is α-dimensional Gillespie measure [Duke Math. J. **6**, 408–419 (1940); these Rev. **1**, 304].]

H. Federer (Providence, R. I.).

Maximoff, Isaiah. On approximately (ρ_n, Δ_n) continuous functions. Tôhoku Math. J. **48**, 34–42 (1941). [MF 16344]

The main purpose is to extend to unbounded functions Denjoy's result that, if f is bounded and approximately continuous at each point of a set H, then, for each $x_0 \in H$,

$$\lim_{|I| \to 0} |I|^{-1} \int_{IH} f(x)dx = f(x_0),$$

where I stands for an interval including x_0. [In an earlier paper [same J. **47**, 237–250 (1940); these Rev. **2**, 352] the author gave the results of the first part of this paper; the proofs given here are still subject to the remarks of the review cited. Moreover, for an approximately continuous function f satisfying (2e) of theorem 1 (this theorem not stated in the previous paper) the result is immediate since each point of its domain H is a point of density of H and (as shown below using the notation of the paper) about each point of H there is an interval I such that f is constant on IH. For, let δ, I, z, n and E_n be a number, interval, point, integer and set, respectively, such that $0<\delta<0.1\rho_1^3$, $m(I)<2\delta$, $m(IH)>0.9m(I)$, $z \in IH$ and E_n is the set of x for which $|f(x)-f(z)|<1/n$. Let $m_1, m_2, \cdots$ be positive integers such that (according to (2e) of theorem 1)

$$m(IH)\{1-2\rho_{m_n}^{-3}m(IH)\}<m(IE_n).$$

Since $P_{m_n}>P_1$, $0.9m(IH)<m(IE_n)$. Let $F_s=\lim_{n \to \infty} E_n$. Thus $0.9m(IH) \leqq m(IF_s)$, $0.81m(I) \leqq m(IF_s)$, and moreover $x \in IF_s$ implies $f(x)=f(z)$. Thus f is constant on IH since, if z' and z'' are in IH, $IF_{s'}$ and $IF_{s''}$ have points in common. Hence each point of density of H may be included in an interval such that f is constant on the part of H in that interval. But if f is approximately continuous at each point of H, then each point of H is a point of density of H.]

In the second part of the paper the terms approximately continuous, summable and Lebesgue integral are replaced, respectively, by a condition (N), totalizable, and Denjoy integral. *J. F. Randolph* (Oberlin, Ohio).

Singh, A. N. On infinite derivates. Fund. Math. **33**, 106–107 (1945).

S. Mazurkiewicz [Fund. Math. **23**, 9–10 (1934)] constructed a function which is continuous on the right and for which $D^+f(x)=+\infty$ everywhere in $[0, 1)$. The author gives another example of this kind, of the utmost simplicity. *A. Rosenthal* (Albuquerque, N. M.).

Gomes, Ruy Luís. The notion of integral based on Jordan measure. Gaz. Mat., Lisboa **7**, no. 29, 5–9 (1946). (Portuguese)

Kline, S. A. On curves of fractional dimensions. J. London Math. Soc. **20**, 79–86 (1945). [MF 16699]

Let the plane curve C be parametrized by the numerically-valued functions x and y on the interval $[0, 1]$, which satisfy Lipschitz conditions of orders p^{-1} and q^{-1}, respectively. Suppose $k=2-p(p^{-1}+q^{-1}-1)$. The author quotes E. R. Love and L. C. Young [Fund. Math. **28**, 243–257 (1936)] to the effect that C has finite k-dimensional Hausdorff measure whenever $p^{-1}+q^{-1}>1$, $1\leqq p\leqq q$, and gives an example in which the Hausdorff dimension of C is exactly k. He further constructs, for $p^{-1}+q^{-1}=1$, $1<p<q$, a space-filling curve C analogous to that of Peano. *H. Federer* (Providence, R. I.).

Walker, A. G. Commutative functions. I. Quart. J. Math., Oxford Ser. **17**, 65–82 (1946). [MF 16773]

A function $f(x)$ is said to be complete in the interval $a\leqq x\leqq b$ if it is continuous and (strictly) monotonic increasing, and if in addition $f(a)=a$, $f(b)=b$. After proving a theorem about a canonical form of complete functions which is essentially due to G. J. Whitrow [Z. Astrophys. **12**, 47–55 (1936)] and after discussing some elementary properties of commutative functions $f(x)$, $g(x)$ (that is, functions for which $f(g(x))=g(f(x))$ the author proceeds to the classification and further investigation of continuous monotonic increasing commutative functions. For this purpose it turns out to be sufficient to consider commutative functions f, g in an interval in which they are both complete and in whose interior points $f(x)\neq x$ [cf. also the following review]. It is shown that there are four mutually exclusive classes of such pairs: rationally related, irrationally related, semirelated and unrelated pairs; pairs belonging to either of the first two classes are called related. The definitions are as follows: f, g are unrelated if there exist integers u, v such that f^u and g^v intersect, that is, have common values for some x without being identical (the upper indices denote the repetition of the functional operation); f, g are related rationally if non-zero integers u, v can be found such that $f^u\equiv g^v$. To establish the remaining definitions it is first shown that in the remaining cases there exists an everywhere dense subset of numbers μ for which the "μth iteration" f^μ of f can be defined (the definition depending on g) and that $L(x)=\lim_{\mu\to 0+}f^\mu(x)$ exists. Then f, g are said to be related irrationally if $L(x)\equiv x$ and semirelated if $L(x)\neq x$. The remainder of the paper is largely concerned with properties, existence theorems and canonical forms of related functions. Finally the notion of pairs of related functions is extended to related sets of a finite or infinite number of functions. In conclusion, some unsolved problems are mentioned. *E. H. Rothe* (Ann Arbor, Mich.).

Walker, A. G. Commutative functions. II. Quart. J. Math., Oxford Ser. **17**, 83–92 (1946). [MF 16774]

This paper deals mainly with semirelated functions [cf. the preceding review]. First the function $L(x)$ of part I is discussed for this case and a similar function $M(x)=\lim_{\mu\to 0+}f^{-\mu}(x)$ is defined. The existence of semirelated functions is proved. New functions L_r, M_r similar to L, M are defined by $L_r(x)=\lim_{\mu\to r+}f^\mu(x)$, $M_r(x)=\lim_{\mu\to r-}f^\mu(x)$ and the relations between these functions and their continuity properties are discussed. Some unsolved problems are mentioned. In an appendix it is shown that there is no real loss in generality when the continuous monotonic increasing functions are assumed to be complete in some interval. *E. H. Rothe* (Ann Arbor, Mich.).

Theory of Functions of Complex Variables

Frazer, H. On functions regular in a convex region. J. London Math. Soc. **20**, 199–204 (1945).

The author proves some new inequalities of the form

$$\int_C |f(z)|^2|dz|\leqq A\int_\Gamma |f(z)|^2|dz|,$$

where $f(z)$ is regular on and inside Γ and C is inside Γ. (a) If Γ is a circle and C is the sum of two circles which touch each other and have radii of Γ as diameters, $A=1$ if $f(0)=0$. (b) If Γ is the strip $-1\leqq x\leqq 1$, $f(z)=o(1)$ uniformly as $|z|\to\infty$, $f(0)=0$ and C is $|z|=1$, $A=1$. (c) If Γ is convex and C is a circle, $A=e+\pi$. The values of A in (a) and (b) are best possible. Result (a) is connected with the inequality

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}{}'\binom{r+s}{r}2^{-r-s-1}a_ra_s\leqq\tfrac{1}{2}\sum_{t=1}^{\infty}a_t^2,$$

where $\sum'$ is restricted to $r+s$ even. The basic theorem of the paper is that, if Γ is a simple closed curve, C is a circle inside Γ, and K is the inverse of Γ with respect to C, then

$$\int_C |f(z)|^2|dz|\leqq\left\{\int_\Gamma |f(z)|^2|dz|\right\}^{\frac{1}{2}}\left\{\int_K |f(z)|^2|dz|\right\}^{\frac{1}{2}}.$$

R. P. Boas, Jr. (Providence, R. I.).

Rosenblatt, Alfred. On the modulus of functions analytic in the unit circle. Actas Acad. Ci. Lima **8**, 27–44 (1945). (Spanish)

For functions $f(z)$ analytic in the unit circle, the author establishes relations between the coefficients in the Taylor expansion of $f(z)$ about the origin to show that, if $|f(z)|$ remains constant on a pair of radii forming an angle α incommensurable with π, then $f(z)$ is identically constant; if, however, α is commensurable with π, then $f(z)$ is not necessarily constant, but $|f(z)|$ must be constant on a set of equally spaced radii. [We note the following inaccuracy in the statement of the theorems: the phrase "se anula" should be replaced by "es una constante." The results might also have been obtained by applying the Schwarz reflection principle.] *E. F. Beckenbach.*

Cinquini, Silvio. Un'osservazione sopra le successioni di funzioni convergenti verso una funzione olomorfa. Boll. Un. Mat. Ital. (2) 4, 29–31 (1942). [MF 16053]

The author strengthens one of his previous theorems [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 17, 502–506 (1933)] on the analyticity of the limit function of a sequence of functions which approximately satisfy the Cauchy-Riemann equations. This is accomplished by weakening the hypotheses which insure the absolute continuity of the limit function. *E. F. Beckenbach.*

Mandelbrojt, Szolem. Sur les fonctions holomorphes et bornées dans un domaine infini. C. R. Acad. Sci. Paris 222, 1157–1158 (1946). [MF 16852]

Mandelbrojt, Szolem, et MacLane, Gérald. Sur les fonctions holomorphes dans une région-bande et une généralisation du problème de Watson. C. R. Acad. Sci. Paris 223, 186–188 (1946).

In an earlier communication [same C. R. 222, 361–363 (1946); these Rev. 7, 378] Mandelbrojt announced the following result. Let $F(s)$ be analytic, bounded and not identically zero in $\Delta: \sigma \geqq 0$, $|t| \leqq g(\sigma)$, where $0 < g(\sigma) \uparrow \pi/2$ as $\sigma \to \infty$; let $A(\sigma) = \int_0^\sigma (\frac{1}{2}\pi - g(u))du$, $A(\infty) < \infty$; then, with $M(\sigma) = \max |F(\sigma+it)|$ for $|t| \leqq g(\sigma)$, $\int_0^\infty \log M(\sigma) e^{-\sigma}d\sigma$ is finite. The first of the notes under review gives a generalization which relaxes the condition $A(\infty) < \infty$. Under certain conditions of regularity on $g(\sigma)$, $\int_0^\infty \log M(\sigma) e^{-\sigma - \alpha A(\sigma)}d\sigma < \infty$ for $\alpha > 1$. The conclusion fails if $\alpha < 1$.

The second note indicates a proof of still more general results, the first of which is as follows. Let Δ_a be the part of Δ for which $\sigma > a$, let $g(\sigma)$ be continuous, positive and of bounded variation in $\sigma > a$, $g(\infty) = \pi/2$; let $S(\sigma) = \frac{1}{2}\pi \int_c^\sigma g(u)^{-1}du$, $c > a$. If $\log |F(\sigma+ig(\sigma)| \leqq -N(\sigma)$, where $N(\sigma)$ increases, then $\int^\infty N(\sigma) e^{-S(\sigma)}d\sigma$ converges. This and similar results are applied to a generalization of Watson's problem: if $|F(s)| \leqq M_n e^{-n\sigma}$, $\liminf M_n^{1/n} < \infty$ or $\lim M_n^{1/n} = \infty$, and $\sum \exp\{-S(\log M_{n+1}^*/M_n^*)\} = \infty$, where $\{\log M_n^*\}$ is the convex regularization of $\{\log M_n\}$, then $F(s) \equiv 0$; this is essentially best possible. *R. P. Boas, Jr.*

Boas, R. P., Jr. A density theorem for power series. Amer. J. Math. 68, 319–320 (1946). [MF 16429]

Let $f(z) = \sum_0^\infty C_n z^n$ have $|z| = 1$ as its circle of convergence. Suppose that $\{C_n\}$ is not bounded but there are M, L and β, $\beta > 1$, such that $|C_{\lambda_n}| \leqq M$ when $|\lambda_n - n\beta| < L$. The author shows that $f(z)$ cannot then be majorized by any integrable function $\psi(\theta)$ in any sector of the unit circle of angle exceeding $2\alpha = 2\pi(1-1/\beta)$; that is, $|f(re^{i\theta})| \leqq \psi(\theta)$, $0 \leqq r < 1$, in any θ-interval of length exceeding 2α, is impossible.

A. C. Schaeffer (Stanford University, Calif.).

Dvoretzky, Aryeh. Sur les changements de signe des coefficients des séries de Dirichlet. C. R. Acad. Sci. Paris 221, 687–689 (1945). [MF 15159]

Let $f(s) = \sum a_n e^{-\lambda_n s}$, $\lambda_n \uparrow \infty$, $s = \sigma + it$, have C and A as the respective abscissas of convergence and absolute convergence. If B is given, $C \leqq B \leqq A$, the author outlines a proof that, for proper choice of ϵ_n, $\epsilon_n = \pm 1$, the series $\sum \epsilon_n a_n e^{-\lambda_n s}$ converges for $\sigma > B$ and has $\sigma = B$ as a natural boundary. The author mentions that this is a generalization of a theorem of Fatou and Pólya for power series, but the known proofs for the power series theorem do not hold in the case of Dirichlet series. *A. C. Schaeffer.*

Augé, Juan. Developments in series of analytic functions and their relations with mixed functionals. Revista Mat. Hisp.-Amer. (4) 6, 3–16 (1946). (Spanish) [MF 16705]

The purpose of this paper is to present a variety of series developments of analytic functions from a common point of view furnished by the theory of mixed linear analytic functionals [L. Fantappié, Teoría de los Funcionales Analíticos y sus Aplicaciones, Barcelona, 1943; these Rev. 7, 308]. The basic series formula is

$$f(x) = F[y(z); x] = \sum_{n=0}^{\infty} y^{(n)}(a) u_n(x)/n!,$$

$$u_{n-1}(x) = \frac{1}{n!}\left[\frac{d^n}{dz^n} u(x, z^{-1}+a)\right]_{z=0} = F[(z-a)^{n-1}; x],$$

$n = 1, 2, \cdots$, $u(x, \alpha)$ being the indicatrix $F[(z-\alpha)^{-1}; x]$. Special choices of the indicatrix are considered; for these various choices the expansion reduces in turn to Taylor's series, the series of Fourier, and an expansion in terms of Legendre functions. Finally, Hilbert's theorem on expansion of an analytic function in a series of polynomials is used to obtain an expansion of $f(x)$ in a series the nth term of which is a polynomial in $u_0(x), \cdots, u_n(x)$. *A. E. Taylor.*

Shen, Yu-Cheng. Interpolation to certain analytic functions by rational functions. Trans. Amer. Math. Soc. 60, 12–21 (1946).

In Verallgemeinerung von Walsh'schen Untersuchungen beweist der Verfasser unter anderem den folgenden Satz. Sei $f(z)$ holomorph für $|z| < 1$ und besitze $f(z)$ die folgende Integraldarstellung,

$$f(z) = (1/\pi) \int\int_K (1-z\bar{t})^{-2} f(t) dS, \qquad |z| < 1$$

(K bedeutet den Einheitskreis); $z = a_{n,i}$, wobei $i = 1, 2, \cdots, n$; $n = 1, 2, \cdots$ sei eine Folge von verschiedenen Punkten in K, die keine Häufungspunkte im Innern von K besitze. Setzt man

$$f_n(z) = \sum_{i=1}^{n} A_{n,i}/(1-\bar{a}_{n,i}z)^2,$$

so ist diese Funktion eindeutig durch die Bedingungen $f_n(a_{n,i}) = f(a_{n,i})$, $i = 1, 2, \cdots, n$, bestimmt. Sofern

$$\lim_{n\to\infty} n \prod_{i=1}^{n} |a_{n,i}|^2 = 0,$$

konvergiert die Folge $f_n(z)$ in jedem innern Bereich von K gleichmässig gegen $f(z)$. *W. Saxer* (Zürich).

Bernstein, Serge. Sur la meilleure approximation sur tout l'axe réel des fonctions continues par des fonctions entières de degré fini. I. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 331–334 (1946).

Bernstein, Serge. Sur la meilleure approximation sur tout l'axe réel par des fonctions entières de degré fini. II. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 487–490 (1946).

An entire function $f(z)$ of finite degree p (also known as a function of exponential type p) satisfies $|f(z)| \leqq Ae^{(p+\epsilon)|z|}$. (1) A necessary and sufficient condition that a function $g(x)$, bounded on the real axis, can be uniformly approximated by functions of finite degree is that $g(x)$ is uniformly continuous. [This has also been proved by a different method by Kober, Trans. Amer. Math. Soc. 54, 70–82 (1943); these Rev. 4, 271.] (2) For a periodic $g(x)$ of period

2π, the best approximation, sup $|g(x)-f_p(x)|$, by functions of degree p is equal to the best approximation by trigonometric sums of order $[p]$. (3) A necessary and sufficient condition for $f_p(x)$, of degree p, to give the best approximation E_p to $g(x)$ is that there exists a positive ϵ such that a function $\varphi_p(z)$ of degree p cannot be bounded on the real axis if it satisfies

$$\varphi_p(t)\{g(t)-f_p(t)\}>0, \qquad |\varphi_p(t)|\geqq 1,$$

at all points t where $E_p-\epsilon\leqq|g(t)-f_p(t)|\leqq E_p$. A further result gives a criterion for the best approximation of degree p to be not less than E_p. As an application, it is shown that $E_p=\frac{1}{2}b^{-2}e^{-pb}(B^2+b^2C^2)^{\frac{1}{2}}$ for $g(x)=(B+Cx)/(x^2+b^2)$. Hence it is shown that, if $g(z)$ is analytic in $|y|<b$ and

$$\int_{-\infty}^{\infty}|g(x\pm bi)|\,dx<M,$$

then lim sup $E_p^{1/p}\leqq e^{-b}$; conversely, if lim sup $E_p^{1/p}=e^{-b}$, $g(z)$ is analytic in $|y|<b$. *R. P. Boas, Jr.*

Morse, Marston, and Heins, Maurice. Topological methods in the theory of functions of a single complex variable. I. Deformation types of locally simple plane curves. Ann. of Math. (2) **46**, 600–624 (1945). [MF 14124]

This paper contains the proofs of results summarized in Proc. Nat. Acad. Sci. U. S. A. **31**, 299–301 (1945) [these Rev. **7**, 57]. The invariance theorem quoted in the review of the summary is proved by admissibly deforming the locally simple curve g first into a piecewise analytic curve by using the theory of conformal mapping and then into a locally simple polygon. The polygon is reduced to a normal form. *M. Shiffman* (New York, N. Y.).

Morse, Marston, and Heins, Maurice. Topological methods in the theory of functions of a complex variable. II. Boundary values and integral characteristics of interior transformations and pseudo-harmonic functions. Ann. of Math. (2) **46**, 625–666 (1945). [MF 14125]

[Cf. the preceding review.] The first part of this paper deals with harmonic functions and their topological generalization, pseudo-harmonic functions. [See Morse, Duke Math. J. **13**, 21–42 (1946); these Rev. **7**, 448, for definitions.] Let M be the number of logarithmic poles of the pseudo-harmonic function U defined over the domain G and let S be the number of saddle points of U in the interior of G counted with their multiplicities. The authors introduce the boundary index I by $M-S=2-\nu+I$, where ν is the number of boundary curves of G, and then evaluate I for specific types of boundary properties of U. For example, if U, as a function on the boundaries of G, has a finite number of relative maxima and minima, then $I=s-m$, where m is the number of points on the boundary which are relative minima of U and s is the number of boundary saddle points of U. Applications of these concepts are made to prove theorems on the number and location of the critical points of harmonic functions.

The second part of the paper is concerned with meromorphic functions and interior transformations. Let $f(z)$ be a meromorphic function or interior transformation in G, and let it be required to analyze the integral characteristics of $f(z)$, namely $n(0)$, the number of zeros of $f(z)$, $n(\infty)$, the number of poles of $f(z)$, and μ, the number of ramification points of the inverse of $f(z)$. The connecting link between this theory and the preceding theory of harmonic functions is obtained by considering the harmonic function $U=\log|f(z)|$. There results the relation $n(0)+n(\infty)-\mu=2-\nu+q-p$, where q is the total order relative to the origin of the images of the ν boundaries of G by $f(z)$ and p is the total angular order of these same images. This relation occurs under the assumption that $f(z)$ is locally simple in the neighborhood of each boundary point of G. If $f(z)$ were analytic on the boundary of G, this relation would be a consequence of

$$\frac{1}{2\pi i}\int\frac{f'(z)}{f(z)}dz=n(0)-n(\infty), \quad \frac{1}{2\pi i}\int\frac{f''(z)}{f'(z)}dz=\mu-2n(\infty).$$

The point of the present theory is lodged in the weaker requirement for $f(z)$ on the boundary of G and the resulting freedom produced by the purely topological analysis required. *M. Shiffman* (New York, N. Y.).

Morse, Marston, and Heins, Maurice. Topological methods in the theory of functions of a single complex variable. III. Causal isomorphisms in the theory of pseudo-harmonic functions. Ann. of Math. (2) **47**, 233–273 (1946). [MF 16333]

[Cf. the two preceding reviews.] The present paper is concerned primarily with an application of the theory of critical points to pseudo-harmonic functions. Consider a function $F(x, y)$ defined on the closure $\bar{G}$ of a finite region bounded by ν Jordan curves $(\beta)=(\beta_1, \cdots, \beta_\nu)$. The function is assumed to be harmonic or pseudo-harmonic on the interior G and to be continuous on the boundary. Moreover, on the boundary no value is assumed more than a finite number of times by F. Let M be the number of logarithmic poles of F, S the number of saddle points of F on G counting these points with their multiplicities, m the number of minimum points and s the number of saddle points of F on the boundary. If M is finite so also are S and s and $M-S=2-\nu+s-m$. If m is not finite, it is denumerably infinite and this relation is replaced by relations between certain groups. These relations are derived from the critical point relations of Morse.

The paper can be divided into three parts. The first is concerned with the definition of saddle points and their multiplicities. The second is concerned with deformation theory. The third part is devoted to the homology theory connecting the theory of critical points. This last part is essentially a redevelopment with simplifications of the theory developed earlier by Morse for a more general situation. [See Morse, Functional topology and abstract variational theory, Mémor. Sci. Math., no. 92, Gauthier-Villars, Paris, 1938.] *M. R. Hestenes* (Chicago, Ill.).

Heins, Maurice. On the number of 1-1 directly conformal maps which a multiply-connected plane region of finite connectivity $p(>2)$ admits onto itself. Bull. Amer. Math. Soc. **52**, 454–457 (1946). [MF 16800]

A conformal automorphism is a 1-1 directly conformal map onto itself of a plane region G of finite connectivity p (>2). It is well known that the number of conformal automorphisms of G is finite and does not exceed $p(p-1)(p-2)$. The author gives the best upper bound $N(p)$ for this number: $N(p)=2p$ if p is different from 4, 6, 8, 12, 20; $N(4)=12$, $N(6)=N(8)=24$, $N(12)=N(20)=60$. He first proves that the group of those automorphisms is isomorphic to one of the finite groups of linear fractional transformations of the extended plane onto itself, by mapping G onto a canonical region G^* whose boundary consists of p circles or points and then extending the conformal automorphisms of G^* to a region Γ bounded by p distinct points. *J. Ferrand.*

Ferrand, Jacqueline. Sur les transformations conformes d'un domaine en lui-même laissant un bout premier invariant. C. R. Acad. Sci. Paris **223**, 189–190 (1946).

Remarks concerning one-to-one directly conformal transformations of a simply-connected region onto itself.

M. H. Heins (Providence, R. I.).

Capelli, Pedro, and Cotlar, Mischa. Some questions related to a possible extension of the principle of conservation of domains. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 63–96 (1946). (Spanish) [MF 16913]

The principle of the conservation of domains is that under conformal transformations interior points are mapped on interior points. For Jordan regions the principle extends, by the Osgood-Carathéodory theorem, to interior points of the boundary. The fact that conformal transformations leave invariant the set property of having zero measure might be considered as an extension of the principle. The authors ask now whether this principle can be extended also to points of density. They obtain an affirmative answer for points of the domain of analyticity but only partial results concerning the boundary.

Applications of some aspects of these considerations are given. Thus the theorem of Fatou states that a harmonic function given by the Poisson integral of a function $u(\theta)$ which is Lebesgue integrable is radially continuous wherever $u(\theta)$ is equal to the derivative of its primitive; this is generalized by replacing Lebesgue integrability by that of Denjoy and derivative by approximate derivative.

E. F. Beckenbach (Los Angeles, Calif.).

Remak, Robert. Ueber eine spezielle Klasse schlichter konformer Abbildungen des Einheitskreises. Mathematica, Zutphen. B. **11**, 175–192; **12**, 43–49 (1943). [MF 15724]

The function $f(z)=z+\sum_2^\infty a_n z^n$ is said to belong to the Hurwitz class if $\sum_2^\infty n|a_n|\leqq 1$. Functions which belong to the Hurwitz class are regular and schlicht in the unit circle $|z|<1$. The author shows that a function of this class maps onto a star-shaped region. Various other properties of this class of functions are discussed, including their behavior on the boundary. The criterion $\sum_2^\infty n|a_n|\leqq 1$ is a "best possible" condition on the moduli of the coefficients in the following sense: if $f(z)=z+\sum_2^\infty a_n z^n$, where $\sum_2^\infty n|a_n|>1$, then for proper choice of the amplitudes of the a_n the function $f(z)$ will not be schlicht in $|z|<1$.

A. C. Schaeffer.

Dvoretzky, Aryeh. Sur une classe de fonctions univalentes. C. R. Acad. Sci. Paris **221**, 605–607 (1945). [MF 15147]

For functions $f(z)=z+a_2z^2+\cdots$ which are regular and schlicht in the unit circle it was shown by Littlewood that $|a_n|<en$. The author shows that if the density of the non-zero coefficients is zero then in fact $a_n=o(n)$. More general results are also stated.

A. C. Schaeffer.

Friedman, Bernard. Two theorems on schlicht functions. Duke Math. J. **13**, 171–177 (1946).

It is proved by elementary methods that, if

$$f(z)=z+a_2z^2+a_3z^3+\cdots$$

is schlicht in $|z|<1$ and has integral coefficients, then $f(z)$ is one of the functions

$$z,\quad \frac{z}{1\pm z},\quad \frac{z}{1\pm z^2},\quad \frac{z}{(1\pm z)^2},\quad z\frac{1\pm z}{1\pm z^2}.$$

[Cf. R. Salem, same J. **12**, 153–172 (1945); these Rev. **6**, 206.] In addition, the estimate $|a_4|\leqq 4.16$ is given for a general schlicht $f(z)$.

W. W. Rogosinski.

Kakeya, Sôichi. On the function whose imaginary part on the unit circle changes its sign only twice. Proc. Imp. Acad. Tokyo **18**, 435–439 (1942). [MF 14769]

Let a function $f(z)=z+c_2z^2+\cdots$ be continuous in the closed unit circle and on the unit circle let its imaginary part change sign only at the two points $e^{i\tau}$ and $e^{i\sigma}$. The coefficients $c_2, c_3, \cdots, c_n$ are shown to lie in a convex domain D_n in real Euclidean $(2n-2)$-space and a supporting curve for D_n is determined. In particular, an explicit expression for the maximum of $|c_n|$ is obtained, the maximum depending on τ, σ and n. In case $\tau=0$ and $\sigma=\pi$, so that $\pm f(z)$ has positive imaginary part on the upper half of the unit circle and negative imaginary part on the lower half, the author obtains the known result that $|c_n|\leqq n$. The inequality $|c_n|\leqq n^2$ is valid for all τ and σ.

A. C. Schaeffer.

Broman, Arne. Sur la convergence des séries potentielles effectuant une représentation conforme d'un cercle sur un domaine borné du plan simple. Ark. Mat. Astr. Fys. **31B**, no. 6, 8 pp. (1944).

The author gives a necessary and sufficient condition for a power series $\sum_{n=0}^\infty a_n z^n$, mapping the circle $|z|<1$ onto a simply connected bounded domain D, to be convergent at every point $z=e^{i\phi}$. As is well known, this can be reduced to topological conditions for the domain. In the author's terminology, it is necessary and sufficient that every chain of transversals is convergent.

[Reviewer's note. This result could be readily proved by reference to the properties of "Primende" and their principal points. The condition for uniform convergence was found by Denjoy, Bull. Soc. Math. France **70**, 97–124 (1942); these Rev. **6**, 207.]

J. Ferrand (Caen).

Golusin, G. On some properties of polynomials. Rec. Math. [Mat. Sbornik] N.S. **18(60)**, 227–236 (1946). (Russian. English summary)

The basis of the paper is the theorem that if $f(z)=\sum_1^\infty c_k z^{-k}$ and $f(z^n+a_1z^{n-1}+\cdots+a_n)=\sum_1^\infty c_k^* z^{-k}$ then the determinant $|A_{rs}^*|$, $A_{rs}^*=c_{r+s-1}^*$, vanishes unless its rank m is a multiple of n $(m=pn)$, when $|A_{rs}^*|=\pm|A_{rs}|^n$, $A_{rs}=c_{r+s-1}$; $r,s=1,\cdots,p$. Using also the connection between these determinants and the transfinite diameter of the set of singularities of the function [G. Pólya, S.-B. Preuss. Akad. Wiss. **1929**, 55–62; Math. Ann. **99**, 687–706 (1928)] it is shown that, if a and b are two numbers not attained by $z^n+a_1z^{n-1}+\cdots+a_n$ when z varies in a domain containing ∞, then the transfinite diameter of its complement is at least $(|a-b|/4)^{1/n}$. A second application is to show that Pólya's inequality for $\limsup |(|A_{rs}|)|^{1/n^2}$ is in general best possible.

A. J. Macintyre (Aberdeen).

Golusin, G. On the problem of Carathéodory-Fejér and similar problems. Rec. Math. [Mat. Sbornik] N.S. **18(60)**, 213–226 (1946). (Russian. English summary)

In the first part of this paper simplified proofs are given for known theorems concerning the class of functions (1) $f(z)=c_0+c_1z+\cdots$ regular in $|z|<1$ and with prescribed coefficients $c_0, c_1, \cdots, c_{n-1}$ [see C. Carathéodory and L. Fejér, Rend. Circ. Mat. Palermo **32**, 218–239 (1911); L. Bieberbach, Lehrbuch der Funktionentheorie, v. 2, Berlin, 1927]. In the second part the author considers the class H_1 of functions (1) regular in $|z|<1$ and such that

$$(1/2\pi)\int_0^{2\pi}|f(re^{i\theta})|\,d\theta\leqq 1.$$

The points with coordinates $(\Re c_0, \Im c_0, \cdots, \Im c_{n-1})$, $f(z)\varepsilon H_1$, fill a closed, bounded region D of a Euclidean $2n$-space. By solving the extremal problem $|\sum_{k=0}^{n-1}\gamma_k c_k|$ = maximum, $f(z)\varepsilon H_1$, the author proves that (i) points on the boundary of D correspond to functions

$$(2)\quad q(z)=(\alpha_0+\alpha_1 z+\cdots+\alpha_\nu z^\nu)^2(\beta_0+\beta_1 z+\cdots+\beta_{n-\nu-1}z^{n-\nu-1})\times(\bar\beta_{n-\nu-1}+\cdots+\bar\beta_0 z^{n-\nu-1}),$$

$\alpha_0+\alpha_1 z+\cdots+\alpha_\nu z^\nu\neq 0$ in $|z|<1$; $0\leqq\nu\leqq n-1$, (ii) if $p(z)=c_0+c_1 z+\cdots+c_{n-1}z^{n-1}$ is not identically zero, then there is a unique polynomial (2) whose Maclaurin series starts with $p(z)$. Among all functions $g(z)=p(z)+c_n z^n+\cdots$ regular in $|z|<1$ this polynomial makes

$$\sup_{r<1}\int_0^{2\pi}|g(re^{i\theta})|\,d\theta$$

a minimum.

A complete solution is given of the problem $|\sum_{k=0}^{n-1}\gamma_k c_k|$ = maximum subject to (1) regular and of absolute value less than one in $|z|<1$. [For partial results see O. Szász, Math. Z. **8**, 303–309 (1920). Some of the author's results were obtained by F. Riesz, Acta Math. **42**, 145–171 (1920).]

W. H. J. Fuchs (Swansea).

Golusin, G. M. On the number of finite asymptotique values of integral functions of finite order. Rec. Math. [Mat. Sbornik] N.S. **18(60)**, 391–396 (1946). (Russian. English summary)

A new proof of the theorem of Ahlfors (Denjoy conjecture), depending, like that of the reviewer [J. London Math. Soc. **10**, 34–39 (1935)], on results of A. Grötzsch. The present proof avoids, however, appeal to the theory of conformal representation of variable domains.

A. J. Macintyre (Aberdeen).

Shah, S. M. A note on the lower order of integral functions. J. Indian Math. Soc. (N.S.) **9**, 50–54 (1945).

Let $f(z)=\sum_0^\infty a_n z^n$ be an entire function; its order ρ and lower order λ are the superior and inferior limits of $(\log\log M(r))/\log r$ as $r\to\infty$. Let $\theta(n)=(n\log n)/\log(1/|a_n|)$. Although $\rho=\limsup\theta(n)$, it is not necessarily true that $\lambda=\liminf\theta(n)$. The author shows, however, that

$$\lambda\geqq\liminf\theta(n)$$

and that $\lambda=\liminf\theta(n)$ if $|a_n/a_{n+1}|$ is nondecreasing for large n. He also points out that λ may fail to equal $\liminf\theta(n)$ when $f(z)$ is of regular growth and that $f(z)$ is not necessarily of regular growth when $|a_n/a_{n+1}|$ is nondecreasing.

R. P. Boas, Jr. (Providence, R. I.).

Shah, S. M. The maximum term of an entire series. II. J. Indian Math. Soc. (N.S.) **9**, 54–55 (1945).

[For part I, cf. Math. Student **10**, 80–82 (1942); these Rev. **4**, 137.] Let $\mu(r)$ be the maximum term of the power series for the entire function $f(z)$, $\nu(r)$ its index. The author proves that, for $0<r_1<r_2$, $(r_2/r_1)^{\nu(r_1)}\leqq\mu(r_2)/\mu(r_1)<(r_2/r_1)^{\nu(r_2)}$.

R. P. Boas, Jr. (Providence, R. I.).

Pisot, Charles. Sur les fonctions arithmétiques analytiques à croissance exponentielle. C. R. Acad. Sci. Paris **222**, 988–990 (1946). [MF 16390]

Pisot, Charles. Sur les fonctions analytiques arithmétiques et presque arithmétiques. C. R. Acad. Sci. Paris **222**, 1027–1028 (1946). [MF 16382]

An "arithmetical function" is a function $f(x)$ such that $f(x)$ is a rational integer when $x=0, 1, 2, \cdots, n, \cdots$. Generalizing results obtained by Pólya, Hardy, Carlson, and Selberg, the author has previously proved the following theorem: there exists a real number $\gamma_0=0.843\cdots$ such that, if $f(x)$ is an entire function satisfying the inequality $|f(x)|<e^{\gamma|x|}$ with $\gamma<\gamma_0$ for large $|x|$, then $f(x)=\alpha_1 P_1(x)+\cdots+\alpha_k P_k(x)$, where the α_i are algebraic integers and the $P_i(x)$ are polynomials. [In particular, if $\gamma<\log 2$, $f(x)=P_1(x)$, which is Pólya's result; if $\gamma<|\log(3+i\sqrt{3})/2|=0.7588\cdots$, then $f(x)=P_1(x)+2^x P_2(x)$, which is an improvement of Selberg's result, and the best possible one; if $\gamma<0.8$, then

$$f(x)=P_1(x)+2^x P_2(x)+\alpha^x P_3(x)+\bar\alpha^x P_4(x),$$

where $\alpha=(3+i\sqrt{3})/2$ and $\bar\alpha$ is its conjugate. Cf. these Rev. **4**, 270.]

The new results of the first paper are (1) the generalization of this theorem to functions not supposed to be entire under the conditions of the regularity of $f(re^{i\phi})$ in an angle $|\phi|<\delta$ ($\delta\geqq\pi/2$) and the regularity of a Laplace transform of f outside a strip parallel to the real axis and of width less than 2π; (2) the application of this result to functions of the form $\sum_0^\infty A_\nu(x)e^{-\lambda_\nu z}$ ($x\geqq 0$), where the A_ν are polynomials of bounded degree. In particular, if a, b are real, $|b|>1$, the function $(a^x-1)/(b^x-1)$ is "arithmetical" if and only if a, b are rational integers and $a=b^m$, m being a positive rational integer.

The main result of the second paper is the establishment of a theorem, analogous to the one stated above, concerning entire functions $f(x)$ taking integral rational values whenever x is a positive or negative integer. *R. Salem.*

Selberg, Henrik L. Über den absoluten Betrag einer analytischen Funktion für gegebene Werte des Funktionsargumentes. Acta Math. **78**, 335–341 (1946).

Let G denote an open unbounded set of the finite z-plane and let $f(z)$ denote a function which is defined, single-valued and analytic for z in G and in addition satisfies $1<|f(z)|<\infty$ for z in G and $|f(z)|=1$ at every finite boundary point of G. Furthermore, let Γ_r denote the intersection of $|z|=r$ with G and let $\lambda(r)$ be defined by

$$\lambda(r)=\int_{\Gamma_r}\{\log|f(re^{i\varphi})|\}^2 d\varphi.$$

In addition, M is to denote a measurable subset of $0\leqq r<2\pi$ of measure $h(M)$, and $\Gamma_r(M)$ the subset of Γ_r for which $\arg f(z)$ is congruent to a value of M (mod 2π). Finally, $\alpha(r, M)$ is defined by

$$\int_{\Gamma_r(M)}\{\log|f(re^{i\varphi})|\}^2 d\varphi=\alpha(r, M)\lambda(r).$$

The following theorem is established. To each δ, $0<\delta<2$, there corresponds an unbounded set Q_δ of r's such that, for $r\varepsilon Q_\delta$, $\liminf_{r\to\infty}\alpha(r, M)\geqq\delta\{h(M)\}^2/16\pi^3$,

$$\liminf_{r\to\infty}\alpha(r, M)\{1-\alpha(r, M)\}\geqq\delta^2\{h(M)\}^2\{2\pi-h(M)\}^2/64\pi^4.$$

The variation of $\log r$ on the subset Q_δ in the interval $1\leqq r\leqq t$ is at least $(1-\frac{1}{2}\delta)\log t-O(1)$. The proof is based on the use of area-length appraisals and the Cauchy-Riemann equations. *M. H. Heins* (Providence, R. I.).

Selberg, Henrik L. Eine Ungleichung der Potentialtheorie und ihre Anwendung in der Theorie der meromorphen Funktionen. Comment. Math. Helv. **18**, 309–326 (1946).

Suppose that a circle cuts a region G along arcs Q with the total angle θ. It is proved that Green's function $g(z, z_0)$

satisfies

$$\int_Q g(z, z_0)d\varphi \leqq \pi^2 \tan \theta/4.$$

With the aid of this lemma a simple proof is given of Collingwood's theorem to the effect that $m(r, a) = O(\log r)$ whenever the Riemann surface W corresponding to a meromorphic function $f(z)$ has only "islands" with a bounded number of sheets over a neighborhood of the point a.

The second fundamental theorem of R. Nevanlinna asserts that the remainder $S(r)$ in

$$\sum_{\nu=1}^{q} m(r, a_\nu)+N(r, 1/f')-N(r, f')+2N(r, f)=2T(r)+S(r)$$

is essentially less than $O(\log T(r))$. A similar estimate in the opposite direction can be expected if W is fairly regularly ramified. The author proves a new inequality in this direction, which goes far beyond known results. His assumptions state that the boundary points of W lie over a finite number of points a_ν and that the algebraic branch-points are not too abundant outside the immediate neighborhoods of these points. *L. Ahlfors* (Cambridge, Mass.).

Selberg, Henrik L. Über eine Ungleichung der Potentialtheorie. Comment. Math. Helv. **18**, 327–330 (1946).

Let $g(z, z_0)$ be the Green's function with pole z_0 of an arbitrary region G. Suppose that a straight line cuts G along a set S of total length L. It is proved that

$$\int_S g(z, z_0)dz \leqq \tfrac{1}{2}\pi L,$$

$\frac{1}{2}\pi$ being the best possible constant. In the proof G is replaced by the majorant slit-region whose Green's function is an Abelian integral. This fact accounts for the possibility of finding explicit estimates.

L. Ahlfors (Cambridge, Mass.).

Kunugui, Kinjiro. Sur l'allure d'une fonction analytique uniforme au voisinage d'un point frontière de son domaine de définition. Jap. J. Math. **18**, 1–39 (1942). [MF 14965]

This paper is concerned with functions which are meromorphic and single-valued in a given region of the complex plane and, in particular, with their behavior in the neighborhood of nonisolated boundary points. The cluster sets introduced by Gross are used systematically. Theorems related to the classical Picard theorem and the Phragmén-Lindelöf principle are obtained as well as theorems on sequences of functions which are meromorphic in the interior of the unit circle and omit three values there.

M. H. Heins (Providence, R. I.).

Tumura, Yosiro. Recherches sur la distribution des valeurs des fonctions analytiques. Jap. J. Math. **18**, 797–876 (1943). [MF 14986]

The contents of the paper are summarized by the author as follows. Dans le chapitre I de cette mémoire nous traiterons principalement le problème de M. Kunugui. En §§ 2 et 3, après l'avoir démontrée aussi précisément que possible, nous construirons un example contraire à sa conjecture. En § 4, après avoir prouvé un lemme sur la représentation conforme, nous étudierons en quel cas sa conjecture se prouve. Le but de chapitre II est d'étudier l'influence d'une classe simple des singularités non-directement critiques sur l'ordre des fonctions. En §§ 5 et 6 nous traiterons par la même méthode l'extension du théorème de MM. Phragmén-Lindelöf et une classe simple des fonctions méromorphes dans tout le plan fini. *M. H. Heins.*

Tumura, Yosiro. Sur le problème de M. Kunugui. Proc. Imp. Acad. Tokyo **17**, 289–295 (1941). [MF 14708]

Tumura, Yosiro. Sur la distribution des valeurs. Proc. Imp. Acad. Tokyo **18**, 612–616 (1942). [MF 14787]

Summary of the paper reviewed above.

M. H. Heins (Providence, R. I.).

***Ghermanescu, Michel. Les combinaisons exceptionnelles des fonctions entières et les fonctions algébroïdes.** Actualités Sci. Ind., no. 889. Hermann et Cie., Paris, 1940. 33 pp.

In the first part the author presents a theory bearing on exceptional linear combinations $\lambda_0+\lambda_1 f_1+\cdots+\lambda_n f_n$ of entire functions, exceptional in the sense that they have only a finite number of zeros. This theory was previously developed by P. Montel [Acta Math. **49**, 115–161 (1926)] and extended by the author [Ann. Sci. École Norm. Sup. (3) **52**, 221–268 (1935)]. In the second part these results are generalized to the case of algebroid functions, and in a note they are tied up with results of H. Cartan [Mathematica, Cluj **7**, 5–31 (1933)].

It is hardly possible to give a detailed account of the results, but it may be noted that the booklet is written without reference to the geometric viewpoint as introduced by H. and J. Weyl [cf. H. Weyl, Meromorphic Functions and Analytic Curves, Ann. of Math. Studies, no. 12, Princeton University Press, 1943; these Rev. **5**, 94].

L. Ahlfors (Cambridge, Mass.).

Laasonen, Pentti. Über die einfachsten zweifach zusammenhängenden Riemannschen Flächen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **9**, 16 pp. (1941). [MF 16499]

The author prescribes the nature of the branch-points of Riemann surfaces F of a class C which, from general results concerning branch-points, can be considered as being the simplest class of doubly-connected Riemann surfaces. A surface F of class C is doubly-connected and of planar character; it has exactly four branch-points, all of which are logarithmic. Two of the branch-points lie over a common point P at which the surface is completely branched. It is shown that such surfaces F exist and a differential equation characterizing the class C is given. *E. F. Beckenbach.*

Laasonen, Pentti. Zum Typenproblem der Riemannschen Flächen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **11**, 7 pp. (1942). [MF 16500]

The author extends results of Ahlfors and Nevanlinna to show that for general Riemann surfaces F the divergence of the integral $\int^{\infty}\{L(\rho)\}^{-1}d\rho$ implies that F has a null boundary. An example is given showing that the converse does not hold. *E. F. Beckenbach* (Los Angeles, Calif.).

Laasonen, Pentti. Beiträge zur Theorie der Fuchsoiden Gruppen und zum Typenproblem der Riemannschen Flächen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **25**, 87 pp. (1944). [MF 16496]

The object of this paper is the study of questions which arise in the transition from the theory of Fuchsian groups to the theory of Fuchsoid groups. In the first part of the paper the author studies the relation between Fuchsoid groups which are of genus zero, free from elliptic trans-

formations, and such that every point of the principal circle is singular, and the associated fundamental polygons. The fundamental properties of such polygons are stated and the converse problem of when a given polygon (with infinitely many sides) in the hyperbolic plane may be regarded as a fundamental domain of a Fuchsoid group is treated. Examples of various cases are given. The singular vertices of a fundamental domain are then studied. The existence of automorphic functions associated with the admitted Fuchsoid groups is treated with the aid of approximation methods which are due to Myrberg and consist in constructing the desired Fuchsoid functions as limits of Fuchsian functions which are automorphic with respect to Fuchsian subgroups of the given Fuchsoid group. Applications are made to the type problem for Riemann surfaces and related questions. *M. H. Heins* (Providence, R. I.).

Dufresnoy, Jacques. Familles complexes quasi-normales. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 287–303 (1946). [MF 16931]

This is a generalization of Montel's theory of normal and quasi-normal families to systems of entire functions (meromorphic curves). There are various applications, leading to theorems of the Schottky-Landau type, etc. *L. Ahlfors.*

Ghika, Al. Sur une extension du théorème de Cauchy et Goursat à certains continus. Bull. Math. Soc. Roumaine Sci. **46**, 13–32 (1944). [MF 16588]

The author gives a neat and complete generalization of the theory of Borel monogenic functions, within the limitations of contour integration. The definitions are as follows. A frontier point of a set, which is a frontier point of at least one component of the complementary set, is a p-point (periphery point); the other frontier points are non-p-points. The periphery Pf (E) of a set E is the set of p-points. A p-point of a closed set is proper if it is a limit point of interior points or of non-p-points. The set of proper p-points is the proper periphery. The main result is as follows. Let K be a continuum with proper periphery Pf (K); assume Pf (K) is rectifiable in the sense that the frontiers of the contiguous domains are rectifiable curves, the sum of whose lengths is finite; then the integral of a function $f(z)$, uniform and having a unique derivative in K, taken along Pf (K) in the same sense with respect to the exterior of K, is zero. A corresponding generalization is given of the integral formula of Cauchy (integration along Pf (K)); this is valid at all points of K having positive angular density. *W. J. Trjitzinsky* (Urbana, Ill.).

Fedoroff, V. S. Sur la monogénéité. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 389–390 (1945). [MF 16652]

For sets of 2^{p-1} complex numbers, which he calls numbers α of rank p, the author defines equality, sum, difference and product. Replacing the components of α by complex functions of n real variables, he then defines integral and partial derivative. For two such functions α, β he next defines monogeneity of α with respect to β; this involves the existence of the limit of a difference quotient, harmonicity, existence of multiplicative inverse, and a conformality condition. The extent of the restrictions implied by these conditions is not necessarily completely determined; but it is stated, among other things, that the derivative of such a monogenic function also is monogenic, that a monogenic function is determined by the values of its derivatives of all orders at a point of monogeneity, and that under certain conditions a Taylor series representation exists. *E. F. Beckenbach* (Los Angeles, Calif.).

Fedoroff, W. Sur la monogénéité. Rec. Math. [Mat. Sbornik] N.S. **18**(60), 353–378 (1946). (Russian. French summary)

The results announced in the preceding paper, and also an analogue of the Cauchy integral formula, are established in detail. *E. F. Beckenbach* (Los Angeles, Calif.).

Special Functions

Montroll, Elliott W. A note on Bessel functions of purely imaginary argument. J. Math. Phys. Mass. Inst. Tech. **25**, 37–48 (1946). [MF 16149]

In this paper the author develops some properties of $I_n(x)=i^{-n}J_n(ix)$. He locates the value x_n which maximizes $e^{-ax}I_n(x)$. He finds, for $\alpha>1$, $x_n\sim n(\alpha^2-1)^{-\frac{1}{2}}$; for $\alpha=1$, $x_n\sim n^2+\frac{1}{2}+O(1/n)$. Furthermore, $\lim_{n\to\infty} I_{n\pm1}(x_n)/I_n(x_n)=\alpha\mp(\alpha^2-1)^{\frac{1}{2}}$, or, writing $\beta=(\alpha^2-1)^{-\frac{1}{2}}$, $\lim I_{n\pm1}(\beta n)/I_n(\beta n)=\beta^{-1}((\beta^2+1)^{\frac{1}{2}}\mp1)$. The author derives four terms of an asymptotic expansion for $I_n(\beta n)$ (n a large integer, $\beta>0$) by applying the method of Laplace for $\int_a^b e^{nf(x)}dx$, where $f(x)$ is a continuous function having a single maximum in the interval (a, b). The first two terms are

$$I_n(\beta n)\sim (2\pi nz)^{-\frac{1}{2}}\{e^z(z-1)/\beta\}^n\{1+(3\beta^2-2)/(24nz^3)+O(n^{-2})\},$$

where $z=(\beta^2+1)^{\frac{1}{2}}$. No upper bound for the remainder or recurrent relation between the consecutive terms is given. [The fact that the main term was obtained by Laplace [Oeuvres, vol. 5, Paris, 1882, p. 489] and several higher order terms by Lehmer [Math. Tables and Other Aids to Computation **1**, 133–135 (1944)] is mentioned, but the results of much wider scope obtained by Debye [Math. Ann. **67**, 535–558 (1909)], van Veen [Math. Ann. **97**, 696–710 (1927)] and Meijer [Math. Ann. **108**, 321–359 (1933)] are ignored.] The above results are applied in discussing the difference-differential equation

$$ay_{n+1}-(a+b+d/dt)y_n+by_{n-1}=0, \qquad n=0, \pm1, \pm2, \cdots,$$

a and b constants, under the boundary conditions

$$y_{-\infty}(t)=y_\infty(t)=0$$

for all t and the initial conditions $y_n(0)=1$ if $n=0$, $y_n(0)=0$ if $n\neq0$. The solution is $y_n(t)=(b/a)^{n/2}e^{-(a+b)t}I_n(2t(ab)^{\frac{1}{2}})$. The function $y_n(t)$ has a single maximum

$$y_n(t_n)\sim\{(b-a)/(2\pi n(b+a))\}^{\frac{1}{2}}.$$

In an appendix Laplace's method is outlined. The result is

$$F(n)=\int_a^b e^{nf(x)}dx\sim (\pi/(zn))^{\frac{1}{2}}e^{nf(\xi)}(1+\lambda_1(nz^2)^{-1}+\lambda_2(nz^2)^{-2}+\cdots);$$

ξ is such that $f'(\xi)=0$; $z=-1/(2f''(\xi))>0$. The rather complicated values of the first three coefficients λ_i are developed. *S. C. van Veen* (Delft).

Banerjee, D. P. On certain integrals and expansions containing Bessel and Legendre's associated functions. Proc. Nat. Acad. Sci. India. Sect. A. **10**, 89–92 (1940).

Applying operational calculus to certain series, the author derives some new integrals and expansions containing

Legendre's associated functions and Bessel functions. For example, from

$$xJ_0(u^{\frac{1}{2}}x)=\sum_{n=0}^{\infty}(4n+2)P_n(1-2u)J_{2n+1}(x),\quad 0\leqq u\leqq 1$$

[Bateman, Messenger of Math. **36**, 31–37 (1906)], he derives

$$x\int_0^1(1-u^2)^{m/2}uJ_m(ux)P_n^m(1-2u^2)du=\frac{2^m(n+m)!}{(n-m)!}J_{2n+1}(x),$$

$$x\int_0^1(1-u^2)^m u^{\frac{1}{2}}J_m(ux)J_{m+\frac{1}{2}}(2ux)du$$

$$=\pi^{\frac{1}{2}}2^{m-1}\sum_{n=0}^{\infty}(4n+2)J_{2n+1}(x)J_{n+\frac{1}{2}}^2(x)x^{-m-\frac{1}{2}}.$$

S. C. van Veen (Delft).

Banerji, D. P. On the expansions and infinite integrals containing Whittaker's M-functions. Proc. Nat. Acad. Sci. India. Sect. A. **11**, 84–86 (1941).

Two expansions of $L_n^{(\alpha)}(x+y)$, one in a series of $L_m^{(\beta)}(y)$, the other in powers of y; the coefficients in each case are expressed in terms of Laguerre polynomials in x. A few infinite integrals containing Laguerre polynomials are evaluated by term-by-term integration. *A. Erdélyi.*

Varma, R. S. An infinite integral involving Whittaker's function. Proc. Nat. Acad. Sci. India. Sect. A. **13**, 40–41 (1943).

Evaluation of the integral

$$\int_0^{\infty}y^{\rho-\sigma-\frac{1}{2}}e^{\frac{1}{2}y}W_{k+n,\rho+n}(y)$$

$$\times{}_3F_2(\nu-2\mu+1;\ \nu+1,\ \nu-2\mu-2n+1;\ -xy)dy$$

in terms of Whittaker functions. *A. Erdélyi.*

Shanker, Hari. On integral representation for Whittaker functions. J. Indian Math. Soc. (N.S.) **9**, 42–46 (1945).

Whittaker's function $W_{k,m}(p)$ has the integral representation

$$W_{k,m}(p)=\frac{e^{-p/2}p^{m+\frac{1}{2}}}{\Gamma(\frac{1}{2}-k+m)}\int_0^{\infty}e^{-px}x^{-\frac{1}{2}-k+m}(1+x)^{m+k-\frac{1}{2}}dx,$$

$$\Re(\tfrac{1}{2}-k+m)>0$$

[see Whittaker and Watson, A Course of Modern Analysis, 4th ed., Cambridge University Press, 1927, § 16.12]. The main theorem given in this paper is as follows. If

$$\phi(x)=e^{-x^2/4}x^{2l-\nu-\frac{1}{2}}W_{k,m}(x^2/2)$$

and

$$\psi(x)=\frac{2^{l-\nu-1}x^{\nu+\frac{1}{2}}\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(\nu+1)\Gamma(l-k+1)}$$

$$\times{}_2F_2\left[\begin{matrix}l+m+\frac{1}{2},\ l-m+\frac{1}{2}\\ \nu+1,\ l-k+1,\end{matrix}\ -x^2/2\right],$$

then $\phi(x)$ and $\psi(x)$ are Hankel transforms of each other of order ν, if $\Re(\nu+1)>0$, $\Re(l\pm m+\frac{1}{2})>0$ and $\Re(l\pm m-\frac{1}{2})>0$.

The proof is based on results by Erdélyi, Goldstein and Varma [Quart. J. Math., Oxford Ser., **10**, 176–189 (1939); Proc. London Math. Soc. (2) **34**, 103–125 (1932); J. Indian Math. Soc. (N.S.) **3**, 25–33 (1938)]. Some special cases involving Kummer's function are also given.

M. A. Basoco (Lincoln, Neb.).

Gupta, H. C. Operational calculus and infinite integrals. Proc. Nat. Acad. Sci. India. Sect. A. **13**, 225–231 (1943).

Evaluation of several infinite integrals with Bessel functions, parabolic cylinder functions, and confluent hypergeometric functions in the integrands, in terms of hypergeometric functions. A sample is

$$\int_0^{\infty}I_\nu(x)K_\nu(x)K_\mu(x\sqrt{2})x^{-\lambda}dx.$$

A. Erdélyi (Edinburgh).

Shastri, N. A. Infinite series involving confluent hypergeometric functions. Proc. Benares Math. Soc. (N.S.) **6**, 11–33 (1944). [MF 16157]

The author discusses series of the type $\sum a_r N_{b_r,\sigma_r}(x)t^r$, where

$$N_{k,m}(x)=\frac{x^{m-\frac{1}{2}}}{\Gamma(2m+1)}M_{k,m}(x)$$

$$=\frac{x^{2m}e^{-x/2}}{\Gamma(2m+1)}{}_1F_1(\tfrac{1}{2}-k+m;\ 2m+1,\ x),$$

$b_r=k+\rho\nu+\alpha r$, $c_r=m+\sigma\nu+\beta r$, $a_r=f(\nu,r)$ in some special cases, for example, $(r+\nu)\Gamma(2\nu+r)/r!$. By using the methods of operational calculus the following theorem is proved. If $h(x,t)\doteqdot pf(p,t)$, where $f(p,t)$ can be expanded in a finite or infinite series of the form $\sum a_r t^r(p-\frac{1}{2})^{(\alpha-\beta)r}(p+\frac{1}{2})^{-(\alpha+\beta)r}$, then

$$\int_0^x N_{k+\rho\nu,m+\sigma\nu}(x-\xi)h(\xi,t)d\xi=\sum a_r t^r N_{k+\rho\nu+\alpha r,m+\sigma\nu+\beta r}(x),$$

where a_r is such that the series on the right is uniformly convergent and h, m, ρ, σ, ν, α and β are such that the integral on the left is convergent and $N_{k,m}$ has a meaning. After discussing the conditions under which the latter series is uniformly convergent, many special cases of this theorem are considered. The author states that the infinite series involving $N_{k,m}$ functions generate as special cases a large number of series involving other members of the family of confluent hypergeometric functions, like Bateman, Laguerre, Sonine and Hermite polynomials, the parabolic cylinder functions, the function $T(m,n)$ introduced by Heatly in the discussion of the Langmuir collector theory, the Bessel function of the third order, etc., as these functions are obtained from the $N_{k,m}$ function for particular values of the parameters k and m. *S. C. van Veen* (Delft).

Feldheim, Ervin. Contributi alla teoria delle funzioni ipergeometriche di più variabili. Ann. Scuola Norm. Super. Pisa (2) **12**, 17–59 (1943). [MF 16766]

The two types of Appell series

$$F_1(a,b,b';c;x,y)=\sum\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}x^my^n,$$

$$F_2(a,b,b';c,c';x,y)=\sum\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_nm!n!}x^my^n$$

are investigated. Using their integral representation, numerous expansions are discussed whose terms contain functions of these types or ordinary hypergeometric functions. Various special and degenerate cases are studied, for instance

$$\phi_2(b,b';c;x,y)=\sum\frac{(b)_m(b')_n}{(c)_{m+n}m!n!}x^my^n.$$

The rest of the paper is devoted to the treatment of the

"Laguerre and Jacobi polynomials" of two variables. The definitions of these polynomials are:

$$L_{m,n}^{(\alpha,\beta)}(x,y)=\frac{(\alpha+1)_{m+n}}{m!n!}\phi_2(-m,-n;\alpha+1;x,y),$$

$$P_{m,n}^{(\alpha,\beta)}(x,y)=$$

$$\frac{(\alpha+1)_{m+n}}{m!n!}F_1\left(m+n+\alpha+\beta+1,-m,-n;\alpha+1;\frac{1-x}{2},\frac{1-y}{2}\right).$$

G. Szegö (Stanford University, Calif.).

Mohan, B. Some infinite integrals. Proc. Nat. Acad. Sci. India. Sect. A. **13**, 171–178 (1943).

In a former paper [Proc. Nat. Inst. Sci. India **7**, 177–182 (1941); these Rev. **5**, 181] the author has studied the properties of the function

$$f_\nu(x)=\sum_{\gamma=0}^{\infty}\frac{(-1)^\gamma x^{\nu+2\gamma}}{2^{\nu+2\gamma}\Gamma(\nu+\gamma+1)},\qquad \Re(\nu)>-1.$$

It was shown that

$$f_\nu(x)=\frac{(x/2)^\nu}{\Gamma(\nu)}\int_0^1(1-u)^{\nu-1}e^{-x^2u/4}du,\qquad \Re(\nu)>0.$$

The object of this paper is to evaluate integrals involving this function. The methods are elementary. The principal results are as follows.

$$(1)\qquad \int_0^\infty x^{p-1}f_\nu(ax)dx=\frac{2^{p-1}\Gamma}{a^p\sin\frac{1}{2}(p+\nu)\pi\Gamma(1-\frac{1}{2}p+\frac{1}{2}\nu)},$$

$\Re(a)>0$, $0<\Re(\nu+p)<2$. The special cases (i) $p=1-\nu$, (ii) $p=1$, (iii) $p=\nu$, (iv) $p=\nu-2$, (v) $p=-\frac{1}{2}$ are noted.

$$(2)\qquad \int_0^\infty x^{-p}f_{\nu+p+1}(ax)J_\nu(bx)dx$$
$$=(a/2)^\nu b^{-1}e^{-\frac{1}{2}b^2/a^2}W_{-\frac{1}{2}\nu-p-\frac{1}{2},\pm\frac{1}{2}\nu}(b^2/a^2),$$

$\Re(a)>0, \Re(b)>0, \Re(\nu+p)\geqq-1, -1<\Re(\nu)<\frac{3}{2}$, where $J_\nu(z)$ is Bessel's function of order ν and $W_{\nu,\mu}(z)$ is Whittaker's function [see Whittaker and Watson, A Course of Modern Analysis, 4th ed., 1927, § 16.12]. The particular cases (i) $p=-1$, (ii) $p=0$, (iii) $p=-\frac{1}{2}\nu-\frac{1}{2}$, (iv) $\nu=1$, (v) $\nu=\frac{1}{2}$ are noted.

$$(3)\qquad \int_0^\infty f_\nu(ax)f_{\nu-1}(bx)dx=\frac{\Gamma(2-\nu)}{(\nu-1)a^\nu}\{(1+a^2/b^2)^{\nu-1}-1\},$$
$$\Re(a)>0,\ \Re(b)>0,\ 0<\Re(\nu)<1.$$

$$(4)\qquad \int_0^\infty xf_\nu(ax)f_\nu(bx)dx=2a^{-\nu}\Gamma(1-\nu)(a^2+b^2)^{\nu-1}.$$

Other particular cases are listed. *M. A. Basoco.*

Sankara Pillai, K. An asymptotic expansion. Math. Student **13**, 82–84 (1945).

The paper is concerned with the asymptotic expansion of the function $f(x)=\sum(x^n/n!)^2$. It may be shown that the series for $f(x)$ is the term independent of t in the product $e^{x/t}I_0(2xt)$, where $I_0(z)$ is the Bessel function of order zero and of purely imaginary argument. It follows that

$$f(x)=\frac{1}{2\pi i}\int_C e^{x/z}I_0(2xz)z^{-1}dz,$$

where C is any circle around $z=0$. For convenience this may be taken as the unit circle. It follows, upon setting $z=e^{iu}$ and using the well-known integral formula for $I_0(z)$, that

$$f(x)=\frac{1}{\pi^2}\int_0^\pi e^{x\cos u}du\int_0^\pi e^{2x\cos\theta\cos u}$$
$$\times\cos(x\sin 2u-2x\sin u\cos\theta)d\theta.$$

A discussion of this formula leads to the result

$$f(x)\sim 3^{-\frac{1}{4}}(2\pi x)^{-\frac{1}{2}}e^{3x}.$$

The complete asymptotic expansion can be found by noting that $f(x)$ satisfies the differential equation $(xd/dx)^3z=27x^2z$. The final result is

$$f(x)\sim 3^{-\frac{1}{4}}(2\pi x)^{-\frac{1}{2}}e^{3x}(1+(9x)^{-1}+2\cdot(9x)^{-2}+\cdots).$$

The author states that this result is of use in statistics.
M. A. Basoco (Lincoln, Neb.).

Difference Equations, Special Functional Equations

Andersen, A. F. Summation of nonintegral order. Mat. Tidsskr. B. **1946**, 33–52 (1946). (Danish) [MF 16306]

This paper treats differences, inverse differences (summation), and difference equations of orders r which are not necessarily integers. When r is real and $a_0, a_1, \cdots$ is a sequence of complex numbers, the sequence $\Delta^r a_0, \Delta^r a_1, \cdots$ of differences (if $r>0$) or inverse differences (if $r<0$) is defined by

$$\Delta^r a_n=\sum_{p=0}^{\infty}\binom{p-r-1}{p}a_{n+p}\equiv\sum_{p=0}^{\infty}(-1)^p\binom{r}{p}a_{n+p},$$

provided the series converge. If $r>0$ and if $\Delta^{-r}a_0$ exists, then $\Delta^{-r}a_n$ exists also for each $n>0$ and the system of equations $\Delta^r x_n=a_n$, $n=0, 1, 2, \cdots$, has the particular solution $x_n=\Delta^{-r}a_n$ and the general solution $x_n=\Delta^{-r}a_n+P(n)$, $n=0, 1, 2, \cdots$, where $P(t)$ is a polynomial in t of degree less than r. *R. P. Agnew* (Ithaca, N. Y.).

Alexiewicz, A., et Orlicz, W. Remarque sur l'équation fonctionelle $f(x+y)=f(x)+f(y)$. Fund. Math. **33**, 314–315 (1945).

Two new short proofs that a measurable solution is necessarily continuous (and hence linear). (1) A measurable solution is approximately continuous at some point and hence, in virtue of the equation, at every point; hence it is of Baire class 1 and so continuous at a point; hence it is continuous everywhere. (2) Let $x\neq 0$ and set $\varphi(t)=f(t)-x^{-1}f(x)t$. Then $\varphi(t)$ has period x and

$$\int_0^x\frac{|\varphi(t)|dt}{\{1+2|\varphi(t)|\}\{1+|\varphi(t)|\}}=0.$$

Hence $\varphi(t)=0$ almost everywhere and $f(t)=f(1)t$ for almost all t; it follows that $f(x)=f(1)x$ for all x. *R. P. Boas, Jr.*

San Juan, R. An application of Diophantine approximations to the functional equation $f(x_1+x_2)=f(x_1)+f(x_2)$. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 221–224 (1946). (Spanish) [MF 16925]

A simple proof that a (finite) solution is either linear or has its graph everywhere dense in the plane. If there exist x_1, x_2 so that $f(x_1)/x_1\neq f(x_2)/x_2$, then, given a and b, we can find rationals r_1 and r_2 so that $r_1x_1+r_2x_2=a+\epsilon$, $f(r_1x_1+r_2x_2)=r_1f(x_1)+r_2f(x_2)=b+\epsilon$; the result follows.
R. P. Boas, Jr. (Providence, R. I.).

The proof was given by Hamel, Math. Ann. 60, 459–462 (1905), p. 462.

Angheluță, Th. Circular transformations characterized by a functional equation. Gaz. Mat., București **51**, 94–98 (1946). (Romanian) [MF 16556]

The equation is

$$3\{f(t+h)-f(t-3h)\}\{f(t+3h)-f(t-h)\} = 4\{f(t+h)-f(t-h)\}\{f(t+3h)-f(t-3h)\}.$$

The only solutions differing from (*) $(at+b)/(ct+d)$ by a continuous function are functions of the form (*).

R. P. Boas, Jr. (Providence, R. I.).

Kreweras, Germain. Sur l'existence des solutions de certaines équations aux itérées. C. R. Acad. Sci. Paris **219**, 303–305 (1944). [MF 15260]

A local study is made of the iterational equation $y=ax+by_2+f(x, y_2)$, where a and b are constants, $y_2=y(y(x))$ and $f(x, y_2)$ is analytic at $(x, y_2)=(0, 0)$ and has a Taylor expansion free from terms of degree less than two. Solutions which are analytic at $x=0$ and vanish there are sought. Sufficient conditions are given with the aid of the theory of normal families. *M. H. Heins* (Providence, R. I.).

Massera, J. L., and Petracca, A. On the functional equation $f(f(x))=1/x$. Revista Unión Mat. Argentina **11**, 206–211 (1946). (Spanish)

To solve the functional equation (1) of the title, the author observes that the transformation $y=1/x$ can be obtained from successive inversions in the unit circle and the real axis; by a linear transformation the x-plane can be mapped on an x'-plane with the circle and axis being carried into the x'-coordinate axes; successive inversions in the coordinate axes of the x'-plane result in the transformation $y'=-x'$. Consequently, the solution of (1) is equivalent to the solution of (2) $f(f(x))=-x$. It is then shown that the general solution of (2) is given implicitly by $F(x^2, f^2)=0$, where F is an arbitrary symmetric function; thus (1) is also solved. *E. F. Beckenbach* (Los Angeles, Calif.).

Mukherjee, Santi Ram. Solutions of the differential equations $f'(x)=f(\pm 1/x)$ where $f(\pm 1/x)$ are properly defined. Proc. Nat. Acad. Sci. India. Sect. A. **12**, 37–45 (1942).

To satisfy the equation of the title the author puts $f(x)=x^m+ax^n$ and obtains an algebraic equation for m together with the conditions $a=m(m-1)\cdots(m-r+1)$, $n=r-m$. The nature of the roots is discussed in some special cases. [For $r=1$ cf. L. Silberstein, Philos. Mag. (7) **30**, 185–186 (1940); these Rev. **2**, 134.] *W. Feller.*

Robinson, Lewis Bayard. Complements to a study of a quasi analytic function which satisfies a functional equation. Actas Acad. Ci. Lima **8**, 129–131 (1945).

[The second word in the title was misprinted "so" in the original.] The author considers the functional equation $u'(x)=\lambda u(x^{-2})/(1-x^2)$. A formal solution is obtained in the form of an infinite series the nth term of which involves an n-fold iterated integral. The series is said to converge for small values of λ. *W. Feller* (Ithaca, N. Y.).

Integral Equations

Zaanen, A. C. On the theory of linear integral equations. I. Nederl. Akad. Wetensch., Proc. **49**, 194–204 = Indagationes Math. **8**, 91–101 (1946). [MF 16573]

The paper contains a generalisation to Hilbert space of the theory of integral equations with symmetrisable kernels. The bounded linear operator K is said to be symmetrisable on the left with respect to the bounded nonnegative definite symmetric operator H if HK is symmetric. It is assumed that the operator $T=EK$ is completely continuous, where E is the projection on the orthogonal complement of the subspace of elements f such that $Hf=0$. The results take their simplest form if $Hf=0$ implies $Kf=0$ and we shall assume this in stating them; when this does not hold the results in general apply to T instead of H.

It is proved that the eigen-values of K are real and that eigen-vectors φ_1 and φ_2 belonging to different eigen-values are H-orthogonal, that is, $(H\varphi_1, \varphi_2)=0$. A form of the minimum-maximum theorem for eigen-values is proved, with $N(f)=(Hf, f)^{\frac{1}{2}}$ playing the rôle of $\|f\|$ and H-orthogonality that of ordinary orthogonality. The expansion theorem takes the form

$$\lim_{k\to\infty} N\left(Kf-\sum_{n=1}^{k}\lambda_n a_n\varphi_n\right)=0,$$

where $a_n=(Hf, \varphi_n)$. It is shown that if $\lambda\neq 0$ the equation $Kf-\lambda f=g$ has a solution f if and only if g is H-orthogonal to all the eigen-vectors of K^* belonging to the eigen-value λ (if any); the solution then satisfies

$$\lim_{k\to\infty} N\left(f+\lambda^{-1}g+\sum_{n=1}^{k}{}'\frac{\lambda_n}{\lambda(\lambda-\lambda_n)}a_n\varphi_n\right)=0,$$

where $a_n=(Hg, \varphi_n)$; the $\sum'$ indicates modifications to be made if λ is an eigen-value.

In the special case $K=AH$, where A is bounded and symmetric, the expansion theorem can be improved; it now becomes

$$Kf=\sum_{i=1}^{\infty}\lambda_i a_i\varphi_i+p,$$

where $Hp=0$. An example is given to show that the term p cannot be omitted in general. A similar result is obtained concerning the expression for the solution of the equation $Kf-\lambda f=g$. *F. Smithies* (Cambridge, England).

Zaanen, A. C. On the theory of linear integral equations. II. Nederl. Akad. Wetensch., Proc. **49**, 205–212 = Indagationes Math. **8**, 102–109 (1946). [MF 16574]

In this paper the author specialises the results of part I [see the preceding review] to the case when K is a symmetric operator of finite norm, which therefore possesses an L^2 kernel, and H is the identity. He thus obtains the standard theorems on integral equations of the second kind with L^2 Hermitian kernels by a method differing from the reviewer's [Proc. London Math. Soc. (2) **43**, 255–279 (1937)]. He also proves that, if $H(x, y)$ is a positive semi-definite continuous kernel, there is an L^2 Hermitian kernel $G(x, y)$ such that $H(x, y)=\int G(x, s)G(s, y)ds$.

F. Smithies (Cambridge, England).

Zaanen, A. C. On the theory of linear integral equations. III. Nederl. Akad. Wetensch., Proc. **49**, 292–301 = Indagationes Math. **8**, 161–170 (1946). [MF 16579]

The author applies the results of parts I and II [see the two preceding reviews] to integral equations of the second kind with kernels of the form $K(x, y)=A(x, y)h(y)$, where $A(x, y)$ is an L^2 Hermitian kernel and $h(x)$ is a bounded nonnegative function. He shows that his generalisation of the theory of symmetrisable kernels [see the review of part I] is completely applicable in this case, without introducing any further restrictions on $h(x)$ (for example, that

$[h(x)]^{-1}$ is bounded). He gives an example showing that the additional term $p(x)$ in the expansion theorem may still arise in this special case. He also remarks that the results can be applied to more general kernels, of the form $k_1(x)A(x, y)k_2(y)$, where $A(x, y)$ is an L^2 Hermitian kernel, $k_1(x)$, $k_2(x)$ and $[k_1(x)]^{-1}$ are bounded and $k_1(x)k_2(x)$ is of constant sign. *F. Smithies* (Cambridge, England).

Sarymsakov, T. A. Généralisation des théorèmes de R. Jentzsch et F. Gantmacher. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 309 (1945). [MF 16656]

The author states without proof two results on integral equations with continuous nonnegative kernels. They concern the existence of positive characteristic functions corresponding to the characteristic value λ_0 of minimum modulus, supposed real, and upper and lower bounds for the value of λ_0. The conditions under which the results are supposed to hold are not quite clear. *F. Smithies.*

Heins, Albert E., and Wiener, Norbert. A generalization of the Wiener-Hopf integral equation. Proc. Nat. Acad. Sci. U. S. A. **32**, 98–101 (1946). [MF 16361]

The authors consider singular integral equations of the form

$$(1) \qquad f(x)=\lambda\int_0^\infty N(x+y)f(y)dy, \qquad x>0,$$

where $N(x)$ is assumed to be integrable in any finite interval excluding $x=0$, $O(e^{-x})$ as $x\to\infty$ and $O(x^{-1})$ as $x\to 0$. They seek solutions that are $O(x^\alpha)$ for a given α such that $-1<\alpha<0$. Equation (1) is rewritten as an infinite sequence of integral equations

$$f_0(x)=\lambda\int_0^\infty \frac{f_0(y)}{x+y}dy,$$

$$f_n(x)=\lambda\int_0^\infty \frac{f_n(y)}{x+y}dy+\lambda\int_0^\infty\left\{N(x+y)-\frac{1}{x+y}\right\}f_{n-1}(y)dy,$$

where $f(x)=\sum_{n=0}^\infty f_n(x)$. The independent solutions of the first equation are known to be of the form x^β; the remaining equations can be solved for $f_n(x)$ in terms of $f_{n-1}(x)$ by known Fourier transform methods. This enables the authors to express $f(x)$ in terms of an infinite series of integral operators acting on a known function. An example is given in which the solution can be obtained in closed form.

The results are stated without proof; there is a confusing misprint in equation (2.4), which should read $f_n(x)=\lambda q_n(x)+\lambda P q_n$. The relation between the numbers d and α in the example is not clear. *F. Smithies.*

Copson, E. T. On an integral equation arising in the theory of diffraction. Quart. J. Math., Oxford Ser. **17**, 19–34 (1946). [MF 15893]

W. Magnus has shown [Z. Physik **117**, 168–179 (1941); these Rev. **4**, 32] that the problem of the diffraction of plane electromagnetic waves by a perfectly reflecting half-plane can be reduced to the solution of the singular integral equation

$$\int_0^\infty f(\xi)H_0^{(2)}(k|x-\xi|)d\xi=2ie^{ikx\cos\alpha}, \qquad x>0.$$

The author shows that the problem of the diffraction of sound waves of small amplitude by a similar obstacle leads to the same equation and solves the equation in the case when k has a negative imaginary part $-q$ and $0<\alpha\leqq\frac{1}{2}\pi$ by using the complex Fourier transform. Under hypotheses implying that $e^{-cx}f(x)\in L(0, \infty)$ for every $c>q\cos\alpha$, he shows that

$$f(x)=ik\sin\alpha\, e^{ikx\cos\alpha}-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{(k-k\cos\alpha)}\sqrt{(k-w)}}{w+k\cos\alpha}e^{-iwx}dw.$$

He states that this solution is still valid when $q=0$. He also derives Sommerfeld's form [Math. Ann. **47**, 317–374 (1896)] for the velocity potential (in the case of sound waves) from his results. *F. Smithies* (Cambridge, England).

Parodi, Maurice. Application du calcul symbolique à la résolution de certaines équations de Fredholm. Bull. Sci. Math. (2) **69**, 174–184 (1945).

The author employs the unilateral Laplace transform to solve integral equations of the forms

$$(*) \qquad \int_0^\infty k(xt)f(x)dx=g(t),$$

$$(**) \qquad \int_0^\infty (x+t)^{-1}k(x+t)f(x)dx=F(t),$$

where the kernels $k(xt)$ in (*) and $k(x+t)$ in (**), $F(t)$ and $g(t)$ are known functions and $f(x)$ is unknown.

A. E. Heins (Pittsburgh, Pa.).

Parodi, Maurice. Application d'une séquence symbolique à la résolution d'équations intégrales. C. R. Acad. Sci. Paris **222**, 1426–1427 (1946).

Richard, Ubaldo. Breve ricerca sulla risolubilità dell'equazione integrale non lineare

$$\varphi(x)+\lambda\int_0^a K(x, y)\varphi^n(y)dy=f(x).$$

Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **78**, 293–311 (1943). [MF 16245]

The author considers the nonlinear integral equations

$$(1) \qquad \varphi(x)+\lambda\int_0^a K(x, y)\varphi^n(y)dy=f(x),$$

$$(2) \qquad \varphi(x)+\lambda\int_0^x K(x, y)\varphi^n(y)dy=f(x),$$

where $n\neq 1$. In case (1), he assumes that $G(x, y)=\int_0^y K(x, t)dt$ is of bounded variation as a function of x, uniformly in y, and is continuous and nondecreasing as a function of y, and that $f(x)$ is a function of bounded variation with a positive lower bound. He proves that, for sufficiently small $|\lambda|$, equation (1) has a unique nonnegative solution $\varphi(x)$ of bounded variation, given by the method of successive approximations; when $\lambda<0$, an additional hypothesis on the upper bound of $\varphi(x)$ is required (as is shown by an example) to ensure uniqueness.

In case (2), he assumes that $K(x, y)$ is continuous in a triangle $0<y<x<a$, except perhaps for singularities of specified type either at the origin or on the line $y=x$, and that $f(x)$ is a continuous function with a positive lower bound. Results broadly similar to those in case (1) are obtained. *F. Smithies* (Cambridge, England).

Solodovnikov, V. V. On an application of operational calculus to dynamical systems with variable parameters. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1945**, 1203–1212 (1945). (Russian)

The author presents an effective method for applying operational calculus to the limiting processes involved in

linear dynamical systems with parameters, depending on time, and in nonlinear dynamical systems. The method is based on the use of successive approximations and of a theorem, determining the operational representation of a product of two functions, with the aid of which one finds the operational representation of the solution of the Volterra integral equation

$$\phi(t)=\phi_0(t)+(d/dt)\int_0^t h(t-\lambda)f(\lambda)\phi(\lambda)d\lambda$$

and of the solution of the nonlinear integral equation

$$\phi(t)=\phi_0(t)+\int_0^t k(t,\lambda,\phi(\lambda))d\lambda;$$

the solution itself is defined by means of the Wagner-Bromwich integral. *W. J. Trjitzinsky* (Urbana, Ill.).

Placzek, G. On the theory of the slowing down of neutrons in heavy substances. Phys. Rev. (2) **69**, 423–438 (1946). [MF 16526]

Although the results of this paper are of interest only to atomic physicists, the mathematical discussion involves the solution of integral equations of rather unusual type. The simplest case (in a notation different from that of the author) is the determination of a function $\phi(x)$ which satisfies the equations

$$(1)\qquad (1-p)x\phi(x)=1+\int_1^x \phi(\xi)d\xi, \qquad 1\leqq x<p^{-1},$$

$$(2)\qquad (1-p)x\phi(x)=\int_{px}^{p^{-1}} \phi(\xi)d\xi, \qquad x>p^{-1},$$

and which is continuously differentiable except at p^{-1} where it has evidently a discontinuity of saltus $-p/(1-p)$.

Equation (1) is equivalent to a first-order differential equation and has the solution

$$\phi(x)=(1-p)^{-1}x^{p/(1-p)}, \qquad 1\leqq x<p^{-1}.$$

Equation (2) is not equivalent to a differential equation and is solved as follows. The interval (p^{-1},∞) is divided into subintervals $I_n:(p^{-n},p^{-n-1})$ so that, when x is in I_n, px is in I_{n-1}. If the function $\phi(x)$ for the interval I_n is denoted by $\phi_n(x)$, (2) becomes

$$(1-p)x\phi_n(x)=\int_{px}^{p^{-n}} \phi_{n-1}(\xi)d\xi+\int_{p^{-n}}^{x} \phi_n(\xi)d\xi.$$

When x is in I_n, make the substitution $z=(q/(1-p))\log p^n x$, $q=p^{1/(1-p)}$, so that the same value of z corresponds to x in I_n and to px in I_{n-1}. In particular, to the upper end of each interval there corresponds the value $Z=-q\log q$. If we change the independent variable by writing

$$\phi_n(x)=(1-p)^{-1}x^{p/(1-p)}\psi_n(z),$$

we obtain

$$e^{z/q}\psi_n(z)=\int_z^Z e^{\zeta/q}\psi_{n-1}(\zeta)d\zeta+q^{-1}\int_0^z e^{\zeta/q}\psi_n(\zeta)d\zeta,$$

from which it follows that $\psi_n'(z)=-\psi_{n-1}(z)$. By Taylor's theorem, we then have

$$\psi_n(z)=\sum_{m=0}^{n}\psi_{n-m}(0)(-z)^m/m!,$$

where the coefficients $\psi_r(0)$ are determined by the condition that $\phi(x)$ is continuous at $x=p^{-n}$ $(n>1)$, so that $\psi_n(0)=\psi_{n-1}(Z)$. It readily follows that $\psi_n(0)=A_n(Z)-qA_{n-1}(Z)$, where

$$A_n(Z)=\sum_{m=0}^{n-1}(m-n)^m Z^m/m!.$$

This completes the solution of the problem.

The behaviour of $\phi(x)$ when $x\gg 1$ is found by considering the behaviour of $\psi_n(z)$ when $n\to\infty$. It is shown by Laplace's method that

$$A_n(Z)\sim\frac{q^n}{1+\log q}+\frac{q^{np}}{1+p\log q}$$

and hence that $\phi_n(x)\sim 1/(1+p\log q)$. *E. T. Copson.*

Functional Analysis, Ergodic Theory, Continuous Geometry

Kantorovitch, L. V. On an effective method of solving extremal problems for quadratic functionals. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 455–460 (1945). [MF 16646]

Let f be an element of a linear metric space and $I(f)$ a quadratic (not necessarily homogeneous) functional. To find an extremum of f the author uses the following method: starting with an arbitrary element $f=f_0$ we determine an element $g=g_1$ which has the same direction as the gradient of $I(f)$ at $f=f_0$, that is, gives for $\|g\|=$constant an extremal value to $[dI(f_0+\alpha g)/d\alpha]_{\alpha=0}$. We then determine a number ϵ_1 in such a way that $I(f_0+\epsilon g_1)$, as a function of ϵ, attains an extremal value for $\epsilon=\epsilon_1$. Put $f_1=f_0+\epsilon_1 g_1$ and apply the same procedure starting from f_1, and so on. The method is illustrated by treating a linear system of algebraic equations, a Fredholm integral equation with symmetric kernel, self-adjoint boundary value problems (linear and partial) and a functional equation of the form $Hf=\varphi$, where H is a positive definite self-adjoint operator in a Hilbert space. In most of these cases it is shown that the convergence to the extremal value of the corresponding functional is as rapid as the convergence of a geometric progression. The author finally remarks that the procedure may also be applied to nonlinear problems (not quadratic functionals) and that the method of the present paper is connected with general concepts regarding extremal problems set forth by him in an earlier paper [same C. R. (N.S.) **28**, 211–214 (1940); these Rev. **2**, 222]. *E. H. Rothe.*

Vulikh, B. Sur l'intégrale de Stieltjes de fonctions, dont les valeurs appartiennent à un espace semiordonné. Leningrad State Univ. Annals [Uchenye Zapiski] **83** [Math. Ser. **12**], 3–29 (1941). (Russian. French summary) [MF 16485]

Let Z be a partially ordered linear space satisfying Kantorovitch's axioms I–V [Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 121–168 (1937)] and possessing a unit element in the sense that $\inf(x,1)>0$ for every $x>0$. The author defines the notions of continuity, uniform continuity, bounded variation and absolute continuity for functions of a real variable taking values in Z. Using his definition of the product of elements of Z [C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 850–854 (1940); these Rev. **2**, 221], he defines the Riemann-Stieltjes integral $\int_a^b x(t)dy(t)$, where $x(t)$ and $y(t)$ both take values in Z, and shows that its most important properties continue to hold.

He then gives a system of 12 axioms for linear spaces with coefficients and norms from Z. These spaces, which he calls "linear normed spaces," include both Banach spaces and partially ordered linear spaces as special cases. He considers distributive operators in such a space taking values in a second one and proves for them the equivalence of continuity and boundedness. If C_Z is the space of uniformly continuous functions from $[a, b]$ to Z with the norm $\sup_{a\leq t\leq b}|x(t)|$, he shows that the general linear operator from C_Z to Z has the form $U(x)=\int_a^b x(t)dy(t)$, where $y(t)$ is a Z-valued function of bounded variation, and works out the explicit form assumed by this theorem in the special case $Z=L$. He also solves the same problem for two other spaces containing the same elements as C_Z, but with different definitions of convergence. *F. Smithies.*

Pinsker, A. G. Universal K-spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 8–11 (1945). [MF 16397]

A Banach space which is partially ordered in the sense of Kantorovitch is called a space of type KB_4 in case (1) the norms of x and $|x|$ are the same, (2) if $|x_1|<|x_2|$ then $\|x_1\|<\|x_2\|$, (3) if $x_n\to 0$ monotonely then $\|x_n\|\to 0$, (4) if $x_n\to\infty$ monotonely then $\|x_n\|\to\infty$. It is called a space of type KB_5 in case (1) to (4) hold and also (5) if $x_1, x_2\geqq 0$ then $\|x_1+x_2\|=\|x_1\|+\|x_2\|$. A subspace X_1 is said to be normally contained in X if for every $x\in X_1$ it is true that $|x'|<x$ implies $x'\in X_1$. With every K-space X is associated its extension $\bar{X}$ in which X is normally contained. The extension of a space of type KB_5 is called a universal K-space. It is stated that every K-space is isomorphic to a part of some universal K-space. It is stated that every space of type KB_5 may be represented as a normed disjunctive sum of spaces with unity. (If X_ξ is of type KB_5, the normed disjunctive sum $\sum X_\xi$ consists of all functions $x_\xi\in X_\xi$ for which $\sum_\xi\|x_\xi\|<\infty$.) The extension of a disjunctive sum is the disjunctive sum of the extensions. Every space of type KB_4 is normally contained in one of type KB_5. Many other results are also stated without proof.

N. Dunford (New Haven, Conn.).

Pinsker, A. G. On the decomposition of K-spaces into elementary spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 168–171 (1945). [MF 16406]

The problem of decomposing an arbitrary K-space [cf. the preceding review] into the direct sum of spaces with simple structure is discussed. It is stated that every K-space X may be decomposed into a direct sum of spaces X_ξ, where X_ξ is the subspace generated by the element x_ξ and the set $\{x_\xi\}$ is an arbitrary complete set of pairwise disjoint non-zero elements of X. An elementary space is a K-space with unit whose extension is of type R_4 [see Kantorovitch, Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 121–168 (1937)]. Properties which characterize elementary spaces are discussed and it is stated that every K-space is a direct sum of elementary K-spaces. The possibility of introducing the operation of multiplication into K-spaces is discussed.

N. Dunford (New Haven, Conn.).

***Ifiiguez Almech, Jose M.: Operadores Lineales en los Espacios Métricos. [Linear Operators in Metric Spaces].** Mem. Acad. Ci. Zaragoza (2) **1**, 273 pp. (1946).

This book is concerned with the basic development of the spectral theory of linear transformations. After introductory chapters on integration and the finite dimensional case, the Riesz-von Neumann theory for the spectral resolution is given. There are two interesting concluding chapters. One of these deals with the space of Wachs, in which quaternions replace the complex numbers. The other discusses nonseparable spaces and makes contact with the theory of almost periodic functions. *F. J. Murray.*

Schatten, Robert, and von Neumann, John. The cross-space of linear transformations. II. Ann. of Math. (2) **47**, 608–630 (1946).

[For part I, by Schatten, see the same vol., 73–84 (1946); these Rev. **7**, 455.] One of the objectives of the development of cross-spaces is the representation of a linear transformation as an element in a linear space. An "expression" $\sum_{i=1}^n f_i\otimes g_i$ does give a transformation T defined by the equation $TG=\sum_{i=1}^n f_i G(g_i)$ from $\mathfrak{B}_2^*$ to $\mathfrak{B}_1$ but all such T's are degenerate, that is, have finite dimensional ranges and do not fill out the completed space. On the other hand, for every linear functional F on $\mathfrak{B}_1\otimes\mathfrak{B}_2$, for f fixed, $F(f\otimes g)=G(g)$ is a linear functional on $\mathfrak{B}_2$. Thus every such F determines a linear transformation A, for which $Af=G$ from $\mathfrak{B}_1$ to $\mathfrak{B}_2^*$ and the bound of F is a "norm" for the transformation A. The above result on F constitutes § 1 of the present paper. The bound λ of T is a crossnorm and § 2 is mainly concerned with the conjugate $(\mathfrak{H}_1\otimes_\lambda\mathfrak{H}_2)^*$ of the λ cross-space of two Hilbert spaces. The trace class of linear transformations A is the set of A's which are products of two Schmidt class operators. For such an A, $|A|=(A^*A)^{\frac{1}{2}}$ has a pure point spectrum with characteristic values λ_i such that $M(A)=\sum_{i=1}^\infty\lambda_i$ is finite. The main result is that $(\mathfrak{H}_1\otimes_\lambda\mathfrak{H}_2)^*=\mathfrak{H}_1\otimes_\gamma\mathfrak{H}_2$, where γ is the greatest cross norm and $M=\gamma$. For Hilbert spaces, then, the conjugation process is clear since it is known that $(\mathfrak{H}_1\otimes_\gamma\mathfrak{H}_2)^*$ is the set of all bounded transformations. For Hilbert spaces, a class of "limited" norms is discussed. These are similar to the associate norm, except for certain dimensional restrictions on the expressions used in defining them. In the appendix, the authors construct a cross norm for n dimensions with the property that $\alpha(I)$ is less than a preassigned positive ϵ. This shows that λ is not the least crossnorm and that there are crossnorms whose associates are not crossnorms.

F. J. Murray (New York, N. Y.).

Kakutani, Shizuo, and Mackey, George W. Ring and lattice characterizations of complex Hilbert space. Bull. Amer. Math. Soc. **52**, 727–733 (1946).

In a previous paper [Ann. of Math. (2) **45**, 50–58 (1944); these Rev. **5**, 146] the authors have shown that, if a real Banach space has a relationship $M\to M'$ between its closed subspaces, with certain of the properties of orthogonal complementation, then X is isomorphic to a real Hilbert space. This is also true if there is a relationship between the transformations on X analogous to taking the adjoint. They now extend these results to infinite dimensional complex Hilbert spaces, using a device of B. H. Arnold to show that an automorphism which takes closed linear sets into closed linear sets must be continuous. Examples are given to show that the situation is different in finite dimensions.

F. J. Murray (New York, N. Y.).

Grinblum, M. M. Sur une classe d'espaces linéaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 467–469 (1945).

Let $\mathfrak{A}$ be a reflexive Banach space of real number sequences such that (1) the elements η_n, where η_n is the sequence whose nth element is 1 and whose other elements are 0's, form an absolute base with the η_n bounded away from zero and infinity, (2) if $a=\{a_i\}$, $b=\{b_i\}$ and $a_i\leqq b_i$ for

each i then $\|a\| \leqq \|b\|$ in both $\mathfrak{A}$ and its conjugate. The author relates this space to the set of sequences $\{x_i\}$, where x_i is an element of the Banach space E and $\{\|x_i\|\} \epsilon \mathfrak{A}$. It is shown that this set $E_{\mathfrak{A}}$ is a Banach space and that separability and reflexivity in E carry over into $E_{\mathfrak{A}}$. If $\xi = \{x_i\} \epsilon E_{\mathfrak{A}}$ and Φ is a linear functional over $E_{\mathfrak{A}}$ then $\Phi(\xi) = \sum_{i=1}^{\infty} F_i^*(x_i)$ with $\{F_i^*\} \epsilon \bar{\mathfrak{A}}$. Conversely, if the latter is true the sum represents a linear functional on $E_{\mathfrak{A}}$. Also $\|\{\|F_i^*\|\}\|_{\bar{\mathfrak{A}}} \leqq Q\|\Phi\|$, where Q is independent of Φ. Necessary and sufficient conditions are given that a set of equations of the form $\sum_{i=1}^{\infty} F_{\alpha i}(x_i) = C_\alpha$ possess a solution in $E_{\mathfrak{A}}$ when E is reflexive and α is any ordinal. Particular applications are given to the case in which E is an L^p space. *R. E. Fullerton.*

Julia, Gaston. Sur les convergences faible et forte dans l'espace d'Hilbert. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 255–272 (1946). [MF 16929]

The author is concerned with various definitions of weak and strong convergence of sequences of points in a Hilbert space. He compiles in one paper several results published previously [C. R. Acad. Sci. Paris **216**, 97–100 (1943); **218**, 376–380 (1944); these Rev. **5**, 186; **6**, 274].

H. H. Goldstine (Princeton, N. J.).

Arnous, Edmond. Sur les groupes continus de transformations unitaires de l'espace de Hilbert: Une extension d'un théorème de M. H. Stone. Comment. Math. Helv. **19**, 50–60 (1946).

This extension has previously been given, using essentially the same methods, by Neumark [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7**, 237–244 (1943); these Rev. **5**, 272], Ambrose [Duke Math. J. **11**, 589–595 (1944); these Rev. **6**, 131], and Godement [C. R. Acad. Sci. Paris **218**, 901–903 (1944); these Rev. **7**, 307].

M. H. Stone (Chicago, Ill.).

Krein, M. On self-adjoint extensions of bounded and semi-bounded Hermitian transformations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 303–306 (1945). [MF 16654]

If the least upper bound $m(T)$ of all γ for which $(Tf, f) \geqq \gamma(f, f)$ is finite the Hermitian transformation T is semi-bounded (from below). Friedrichs has proved the conjecture of von Neumann which asserts that if the domain of the semi-bounded Hermitian transformation T is dense then there is at least one semi-bounded self-adjoint extension $\bar{T}$ of T for which $m(\bar{T}) = m(T)$. The present paper solves the problem of determining all self-adjoint extensions $\bar{T}$ of T for which $m(\bar{T}) \geqq \gamma$, where $\gamma \leqq m(T)$ is an arbitrary real number. Among the self-adjoint extensions $\bar{A}$ (with $|\bar{A}| \leqq 1$) of a bounded operator A (with $|A| \leqq 1$, $D(A)$ closed and nondense) are two extreme ones A_μ and A_M such that a bounded self-adjoint operator B is such an extension of A if and only if $A_\mu \leqq B \leqq A_M$. Analogous results are given for the case of unbounded operators with dense domain. A number of extremal properties for these "extreme" extensions are given. *N. Dunford* (New Haven, Conn.).

Frola, Eugenio. Un teorema sulle iterazioni successive dei nuclei continui simmetrici e omogenei di grado I agenti in L_2. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **76**, 468–470 (1941). [MF 16265]

Let N be a bounded symmetric operator in Hilbert space and consider the iteration process $y_{n+1} = Nx_n$, $x_{n+1} = y_{n+1}/\|y_{n+1}\|$. If the sequence $\{\|y_n\|^{-1}\}$ is convergent to a nonzero limit λ, and the set of limit points of the sequence $\{x_n\}$ is nonvoid, then every such limit point is an eigenvector of N^2 with eigen-value λ^2. *F. Smithies.*

Fullerton, R. E. Linear operators with range in a space of differentiable functions. Duke Math. J. **13**, 269–280 (1946).

This paper gives generalizations to the case $C^n(0, 1)$ of known results for $C(0, 1)$. The main contribution consists of two theorems on linear transformations from the Banach space E to $C^n(0, 1)$ and applications to the cases that E is l_p, c, L_p. The general result is that now n families of functionals are associated with a linear transformation instead of one family. [It seems to the reviewer that there is a gain in transparency by the introduction of the conjugate operation. Thus, introduce the functionals (over C^n)

$$\{\psi_T^i \mid 0 \leqq T \leqq 1;\ i = 0, \cdots, n\}$$

by defining $\psi_T^i y = d^i y(t)/dt^i$ at $t = T$. It is then well known that these are elements of C^{n*}. A linear transformation T on E to C^n associates the conjugate operator T' on C^{n*} to E^*. Write $f_T^i = T'\psi_T^i$. Plainly, $f_T^i x = d^i(f_T^0 x)/dT^i$ and $\|Tx\| = \sup_{i,T} |\psi_T^i y| = \sup_{i,T} |f_T^i x|$, whence $\|T\| = \sup_{i,T} \|f_T^i\|$.] The assertion of the existence of such functionals f_T^i constitutes the author's first main theorem (theorem 2). His second (theorem 3), as in the parallel situation for $n = 0$, states that $f_T x \epsilon C^n$, $0 \leqq T \leqq 1$, determines the n families $\{f_T^i\}$, with $f_T^0 = f_T$, and a linear transformation T.

D. G. Bourgin (Urbana, Ill.).

Kondrachov, W. Sur certaines propriétés des fonctions dans l'espace. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 535–538 (1945). [MF 16643]

The author states theorems of the following type. If L_p^r denotes the Banach space of functions in a domain of n variables which, together with their partial derivatives of order at most r, have an L_p-norm, then the unit sphere of L_p^r is compact in L_q^{r-k}, where $q < 1/(p^{-1} - kn^{-1})$. The derivatives need not be literal derivatives but may be defined by adjointness [S. Soboleff]. The author also has a conclusion from boundedness on the boundary of a domain to boundedness inside the domain and other partial improvements on similar theorems which are customary tools in potential and operator theory. *S. Bochner.*

Siddiqi, Raziuddin. Sur les espaces hilbertiens hypercomplexes. C. R. Acad. Sci. Paris **222**, 1469–1470 (1946).

The author defines a hypercomplex Hilbert space as an infinite-dimensional complete separable linear space with an inner product whose values are complex numbers. Multiplication of the complex numbers is in general noncommutative; each number has a conjugate, with which it commutes, and the product of a number and its conjugate is real and nonnegative. The metric of the space is then introduced as in ordinary abstract Hilbert space. This note announces that many of the concepts and results of ordinary Hilbert space can be extended to hypercomplex Hilbert space, and refers to the author's writings on the subject in J. Osmania Univ. Hyderabad. *A. E. Taylor.*

Monna, A. F. On non-Archimedean linear spaces. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **52**, 308–321 (1943). (Dutch. German, English and French summaries) [MF 15936]

Let K be a field complete with respect to a non-Archimedean valuation $|\cdots|$, in which the values, other than 0, form an ordered multiplicative Abelian group P in which every set bounded above has a least upper bound. Let E be a vector space over K complete with respect to a norm $\|\cdots\|$

such that (1) if $x \in E$, $x \neq 0$, then $\|x\| \in P$, (2) $\|ax\| = |a| \cdot \|x\|$, $a \in K$, (3) $\|x+y\| \leqq \max(\|x\|, \|y\|)$. Analogues to various results in ordinary Banach spaces are proved, for example, the Hahn-Banach theorem and the theorem on the continuity of the inverse of a one-to-one continuous linear operator. [In some of the proofs, the completeness of E or even of K may be dispensed with.] Applications are made to certain spaces of sequences and to the solution of linear equations over K in infinitely many unknowns. [The author seems to be tacitly assuming throughout that P has a cofinal countable subset. Although he does not justify this assumption in the present paper [see the second following review], it is true that the conditions imposed on P imply that it is either the multiplicative group of all positive real numbers or one of its cyclic subgroups. Some corrections to this article are made in the paper reviewed below.]

I. S. Cohen (Philadelphia, Pa.).

Monna, A. F. Linear functional equations in non-Archimedean Banach spaces. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **52**, 654–661 (1943). (Dutch. German, English and French summaries) [MF 16047]

With the notation as in the preceding review, let it be assumed that K is locally compact; P must then be cyclic. The conjugate space is introduced and some properties of continuous linear functionals are studied. Many of the properties of completely continuous operators in the classical case hold also in the non-Archimedean case.

I. S. Cohen (Philadelphia, Pa.).

Monna, A. F. On ordered groups and linear spaces. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **53**, 178–182 (1944). (Dutch. German, English and French summaries) [MF 16321]

Let P be an ordered group. Unnecessarily long proofs are given of the facts that (1) if every bounded subset of P has a least upper bound, then P is Archimedean; (2) if P is Archimedean and has a least element greater than the unit element, then P is cyclic. With reference to the paper reviewed before the preceding one, the possibility of relaxing the conditions on P and K is discussed. *I. S. Cohen.*

Monna, A. F. On the integral of a function whose values are elements of a non-Archimedean valued field. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **53**, 385–399 (1944). (Dutch. German, English and French summaries) [MF 15938]

Let K be as in the third preceding review. Let G be a compact topological space; let M be the non-Archimedean Banach space of all bounded functions from G to K, with least upper bound as norm; let C be the subspace of all continuous functions. An integral for continuous functions is considered to be defined by any fixed continuous linear functional F on C. This functional can be extended to M [cf. the review mentioned above] and, in particular, to characteristic functions of subsets of G. There is thus defined a measure, with values in K, for subsets of G, and F is uniquely determined by this measure. Conversely, any finitely additive and bounded set function on G defines an integral. The derivative (in an appropriate sense) of the integral of a continuous point function over G is equal to the value of the function at all points except on a set of which every perfect subset has measure zero. As an application it is shown that there is no nontrivial p-adic finitely additive measure on the set of p-adic numbers which is defined for all closed sets and is invariant under translation.

I. S. Cohen (Philadelphia, Pa.).

Munroe, M. E. A second note on weak differentiability of Pettis integrals. Bull. Amer. Math. Soc. **52**, 668–670 (1946).

Let E be a compact metric space and $C(E)$ the space of continuous functionals on E. In a preceding note [same Bull. **52**, 167–174 (1946); these Rev. **7**, 307] the author proved that if E is uncountable there is a function from the unit interval to $C(E)$ whose Pettis integral fails on a set of positive measure to have a weak derivative. In this note it is shown that such a function exists if E is only infinite, not necessarily uncountable. It follows that every Pettis integral from the unit interval to an abstract M-space is almost everywhere weakly differentiable if and only if the M-space is finite dimensional. *M. M. Day.*

Ríos, S. Note on analytic operations in Banach spaces. Revista Mat. Hisp.-Amer. (4) **6**, 48–50 (1946). (Spanish) [MF 16709]

The author states without proof that, if $f(z)$ is a function on the complex plane to a complex Banach space B, and if $f(z)$ is analytic in a domain D, then $f(z)$ is the limit of a sequence of polynomials (with coefficients in B) which converge uniformly "in the interior" of D. [Clearly, the additional assumption that D is simply connected must be made.] The author then defines polynomials with arguments in a Banach space and proposes that a function with arguments and values in complex Banach spaces shall be called analytic in a domain if it admits a sequence of polynomials converging to it in the above manner. The author promises to give details of the development of this definition elsewhere. *A. E. Taylor* (Los Angeles, Calif.).

Hyers, D. H. A generalization of Fréchet's differential. Revista Ci., Lima **47**, 645–663 (1945).

The author is concerned with functions whose arguments and values lie in linear topological spaces. For such functions he proposes a definition of a differential. The choice of the definition is guided in part by the following requirements: (a) for normed linear spaces the definition must be equivalent to that of the Fréchet differential; (b) the theorem about the differential of a function of a function must be valid; (c) the differential must be unique; (d) the differential must be invariant under topological isomorphisms of the spaces involved; (e) there must exist a nontrivial example of a differentiable function with arguments and values in nonnormable linear topological spaces.

A pseudo-norm of an element x is a functional of x and an element d of a strongly partially ordered space D [Hyers, Duke Math. J. **5**, 628–634 (1939); these Rev. **1**, 58]. It is denoted by $|x|_d$. Let L_1 and L_2 be linear topological spaces with pseudo-norms associated with strongly partially ordered spaces D_1 and D_2, respectively. A function f defined in a neighborhood of x in L_1, with values in L_2, is said to have an F-differential $f(x;h)$ at x if (1) $f(x;h)$ is an additive and continuous function of h on L_1 to L_2 and (2) to each e in D_2 there corresponds a d in D_1 such that for every positive η there exists a positive δ for which $|h|_d < \delta$ implies $|f(x+h)-f(x)-f(x;h)|_e \leqq \eta |h|_d$. This definition fulfills requirements (a)–(e). It is proved that a differentiable function is continuous and that the Gâteaux differential exists and coincides with the F-differential if the latter exists. The paper concludes with a discussion of the relation of the

F-differential to the M- and M_1-differentials, as defined by A. D. Michal. If f has an F-differential, it also has an M-differential [Michal, Proc. Nat. Acad. Sci. U. S. A. **24**, 340–342 (1938)] and the two are equal. On the other hand, f need not have an M_1-differential [as defined by Michal, Revista Ci., Lima **47**, 389–422 (1945); these Rev. **7**, 308]. *A. E. Taylor* (Los Angeles, Calif.).

Gottschalk, W. H. A note on pointwise nonwandering transformations. Bull. Amer. Math. Soc. **52**, 488–489 (1946). [MF 16807]

A continuous transformation f of a T_1-space X into itself is called pointwise nonwandering if each neighborhood U overlaps infinitely many of its images $f^n(U)$. It is shown (1) that, if f is pointwise nonwandering, then so also is f^k for every positive integer k and (2) that, if X is connected and $f(X)=X$ is a pointwise nonwandering homeomorphism, then each cut point of X is periodic. *J. C. Oxtoby.*

Gottschalk, W. H. Almost periodicity, equi-continuity and total boundedness. Bull. Amer. Math. Soc. **52**, 633–636 (1946).

Let X be a uniform space, let T be a topological group, and let $f(x, t)=f^t(x)$ be a mapping of $X\times T$ into X which defines a homomorphism $t\to f^t$ of T onto a group G of transformations of X. A set $E\subset T$ is called relatively dense if there exists a compact set $A\subset T$ such that each left translate of A intersects E. A point $x\in X$ is called almost periodic (a.p.) if to each neighborhood index α corresponds a relatively dense set $E\subset T$ such that $f(x, E)\subset U_\alpha(x)$. If the set E can be chosen independently of x, then the group G is called a.p. The principal results are as follows. If the family G is equicontinuous at an a.p. point x, and $f(x, t)$ is continuous in t for this x, then the orbit of x is totally bounded. If the family G is equiuniformly continuous and the orbit of x is totally bounded, then x is a.p. If X is compact and f is continuous on $X\times T$, then the following statements are equivalent: (1) G is a.p., (2) G is equiuniformly continuous, (3) G is totally bounded in the space of transformations of X. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Gottschalk, W. H., and Hedlund, Gustav A. Recursive properties of transformation groups. Bull. Amer. Math. Soc. **52**, 637–641 (1946).

Consider a group of transformations of a topological space X. The points of X can be classified in various ways according to the behavior of their orbits. In this paper there is defined a general notion of recursive point and it is shown that if a point is recursive with respect to the given group then it is recursive with respect to certain subgroups. By appropriately specializing this result, in case the given group is a discrete or continuous flow, it follows that whenever a point is, for instance, almost periodic, recurrent or strongly recurrent with respect to the given flow, then it has the same property with respect to any discrete subflow. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Barbachine, E. Sur la conduite des points sous les transformations homéomorphes de l'espace. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 3–5 (1946).

This paper outlines a generalization of the general theory of Birkhoffian dynamics [cf. Dynamical Systems, Amer. Math. Soc. Colloquium Publ., v. 9, New York, 1927, chapter 7] to the case where the underlying parameter space is a partially ordered group instead of the real axis. The group G and the ordering relation are assumed to satisfy the following conditions. (1) (a) It is not true that $g>g$; (b) if $g_1>g_2$ and $g_2>g_3$ then $g_1>g_3$. (2) If $g_1>g_2$, then $gg_1>gg_2$ and $g_1g>g_2g$ for all g in G. (3) There exists an infinite sequence (called an ω-sequence) $g_1<g_2<g_3<\cdots$ of elements of G such that, corresponding to any g in G, there exists a g_n in this sequence such that $g<g_n$. (The inverses of an ω-sequence form an α-sequence.)

Now let G act on a compact metric space X in the sense that (A) corresponding to x in X and g in G there exists a unique point $g(x)$ of X; if e is the identity in G, $e(x)=x$; (B) if $g_1\in G$, $g_2\in G$, and $x\in X$, then $(g_1g_2)(x)=g_1(g_2(x))$; (C) if $x_n\in X$ and $x_n\to x$, then $g(x_n)\to g(x)$ for all g in G. The point x of X is wandering if there exists a neighborhood $U(x)$ of x and an element g_0 of G such that $g>g_0$ implies $g(U(x))\cap U(x)=0$. As in the case of a flow, it now follows that a set of central motions (the center) can be defined by means of a well-ordered decreasing set of sets. The point x is said to be stable if, for any $U(x)$ and g_0 of G, there exists a g in G such that $g(x)\in U(x)$ and $g>g_0$. The stable points are everywhere dense in the center.

If $g_1<g_2<\cdots$ is an ω-sequence and $g_n(x)\to y$, y is said to be an ω-limit point of x. The α-limit points are defined analogously. If minimality is defined in the usual manner, it is stated that the ω-limit or α-limit set of any point x contains a minimal set. The point x is recurrent if corresponding to $\epsilon>0$ there exists a g_ϵ in G such that, if y_1 and y_2 belong to the trajectory of x, an element g of G can be found such that $e\leqq g\leqq g_\epsilon$ and $\rho(g(y_1), y_2)<\epsilon$. Every point of a minimal set is recurrent. In order that the closure of the orbit of a recurrent point be minimal, it is necessary to replace (C) by a stronger hypothesis. *G. A. Hedlund.*

Denjoy, Arnaud. Les trajectoires à la surface du tore. C. R. Acad. Sci. Paris **223**, 5–8 (1946).

The author has previously shown [J. Math. Pures Appl. (9) **11**, 333–375 (1932)] that, if a topological sense-preserving transformation f of a circle C onto itself is such that the derivatives of f and f^{-1} are of bounded variation and no power of f has a fixed point, then f is transitive in the sense that the orbit under powers of f of every point of C is everywhere dense on C. In the present paper a simplified proof of this fact is derived. If J denotes the closure of the orbit of a point of C then either $J=C$ or J is a discontinuum. If the latter case occurs, it is shown that, if i_0 is an interval of $C-J$, and $i_n=f^n(i_0)$, then $\log(i_n/i_0)$ is bounded, which yields a contradiction. The proof makes use of continued fractions. *G. A. Hedlund* (Charlottesville, Va.).

Halmos, Paul R. An ergodic theorem. Proc. Nat. Acad. Sci. U. S. A. **32**, 156–161 (1946). [MF 16712]

An ergodic theorem of wide generality is proved. The underlying space X is a measure space which is the union of countably many measurable sets of finite measure. The transformation T is a one to one transformation of X onto X such that T is measurable (T and T^{-1} send measurable sets into measurable sets) and nonsingular (T and T^{-1} send sets of measure zero into sets of measure zero). Under these conditions it is known that there exists a measurable function $\omega_n(x)$, positive almost everywhere, such that $m(T^nE)=\int_E\omega_n(x)dm(x)$. If $q(x)$ is a real valued function defined on X, and n is a positive integer, let $q^n(x)=\sum_{i=0}^{n-1}q(T^ix)\omega_i(x)$. The theorem asserts that, if $f(x)$ is integrable and $g(x)$ is nonnegative, measurable and such that $\lim_{n\to\infty}g^n(x)=\infty$ almost everywhere, then $f^n(x)/g^n(x)$ converges almost everywhere to a finite limit.

The Birkhoff ergodic theorem is immediately obtained as a special case of the preceding theorem. It is shown that the Hopf and Hurewicz ergodic theorems follow from the stated theorem and easily derived additional results. The proof of the theorem employs techniques similar to those which have been used by Khintchine [Math. Ann. **107**, 485–488 (1932)], E. Hopf [Ergodentheorie, Ergebnisse der Math., v. 5, no. 2, Springer, Berlin, 1937, p. 49] and Hurewicz [Ann. of Math. (2) **45**, 192–206 (1944); these Rev. **5**, 148]. The author shows that in the incompressible case (for every measurable set E, $TE \subset E$ implies $m(E-TE)=0$) his theorem would be an easy consequence of Hopf's theorem, if it could be shown that the nonsingularity of T implies the existence of a measure m^* invariant under T, such that $m^*(E)=0$ if and only if $m(E)=0$ and such that X is the union of countably many sets of finite m^* measure. However, the existence of such an invariant measure is one of the unsolved problems connected with this theory.

G. A. Hedlund (Charlottesville, Va.).

Riesz, Friedrich, und v. Sz. Nagy, Béla. Über Kontraktionen des Hilbertschen Raumes. Acta Univ. Szeged. Sect. Sci. Math. **10**, 202–205 (1943). [MF 16747]

A new proof of the mean ergodic theorem for a linear contraction in Hilbert space is given. The proof differs from an earlier one given by F. Riesz [J. London Math. Soc. **13**, 274–278 (1938)] in that it avoids explicit use of the fact that bounded sequences are weakly sequentially compact. This is accomplished by means of a theorem asserting that a linear contraction in Hilbert space has the same manifold of invariant elements as its adjoint. *N. Dunford.*

Iwamura, Tsurane. On continuous geometries. I. Jap. J. Math. **19**, 57–71 (1944). [MF 14995]

Let L be a continuous geometry, in general reducible. Since the centre Z is a Boolean algebra its elements can be represented as the open-and-closed subsets of a suitable bicompact totally disconnected space S. For $a \in L$, $p \in S$, the author defines the congruence class a/p as the class of x for which $x \in L$ and $ex=ea$ for some e such that $p \in e$, $e \in Z$. The a/p form a complemented modular lattice L/p in which perspectivity is transitive and any two elements are (perspectively) comparable. If L is of a type $k=1, 2, \cdots, \infty$, a dimension function $\delta(a)=\delta(a/p)$ can be defined with $0 \leqq \delta \leqq 1$, $\delta(a+b)+\delta(ab)=\delta(a)+\delta(b)$, and for $e \in Z$, $\delta(e)=1$ or 0 according as $p \in e$ or $p \bar{\in} e$. The $\delta(a)$, as functions of p, form a complete lattice D of functions continuous on S; $\delta(a) < \delta(b)$, $\delta(a)=\delta(b)$ or $\delta(a) > \delta(b)$ according as $a \dashv b$, $a \sim b$ or $a \vdash b$; $\delta(a)$ is unrestrictedly additive and continuous in the sense of (o)-convergence in both L and D. Then $\overline{a/p}$ is defined as the class of x in L with $\delta(a+x/p) \doteq \delta(ax/p)$. The $\overline{a/p}$ form an irreducible continuous geometry $\overline{L/p}$ ($=L/p$ if $k < \infty$) with dimension function $\delta(\overline{a/p})=\delta(a/p)$. The representation theorem follows: $a \leftrightarrow \sum \oplus \overline{a/p}$, $p \in S$, maps L lattice-isomorphically on L', a sublattice of the direct sum $\sum \oplus \overline{L/p}$, $p \in S$. For an element $\sum \oplus x_p$, $p \in S$, to be in L', $\delta(x_p)$ must be continuous in p, but the problem remains to complete the characterization of L'. Finally, the dimension function and the representation theorem are obtained for a general L, using a theorem of J. von Neumann [Continuous Geometry, Part III, Institute for Advanced Study, Princeton, N. J., 1937] that L is a direct sum $\sum_{i \leqq \alpha} \oplus L_i$ of geometries of type k. The dimension function is used to show that perspectivity in L is unrestrictedly additive and continuous, a result obtained by the reviewer [Duke Math. J. **5**, 503–511 (1939); these Rev. **1**, 30]. [In the footnote, p. 59, $g(p)$ should be defined as the number in $F=\underline{\mathrm{bd}}\ (s_n; p \bar{\in} M_n)$.]

I. Halperin (Kingston, Ont.).

Kawada, Yukiyosi, Higuti, Kaneo, und Matusima, Yatarô. Bemerkungen zur vorangehenden Arbeit von Herrn T. Iwamura. Jap. J. Math. **19**, 73–79 (1944). [MF 14996]

The representation theorem of Iwamura [see the preceding review] is obtained in terms of maximal p-ideals; I is a maximal p-ideal in L if it is maximal with the properties that $a \in I$ and $b \in I$ imply $a+b \in I$, and $c \dashv a$ or $c \sim a$, $a \in I$, imply $c \in I$. Then $P(I)=I \cap Z$ is a maximal ideal in Z and conversely every maximal ideal in Z is of this form with a unique I. Call $d(a)$ an elementary dimension function in L if $0 \leqq d(a) \leqq 1$, $d(0)=0$, $d(1)=1$, $d(a+b)+d(ab)=d(a)+d(b)$, and, for every $e \in Z$, $d(e)=0$ or 1. Then $I(d)$, consisting of those $a \in L$ for which $d(a)=0$, is a maximal p-ideal and $L/I(d)$ is an irreducible continuous geometry. If L is of a type $k=1, 2, \cdots, \infty$, then every maximal p-ideal is of this form and the representation theorem follows. The geometry $L/I(d)$ is the $\overline{L/p}$ of Iwamura, where p is the point corresponding to $P(I(d))$ in the point-set representation of Z.

I. Halperin (Kingston, Ont.).

Theory of Probability

Singh, Jagjit. Theories of probability. Sankhyā **7**, 257–262 (1946).

Let f_n be the frequency of successes in the first n trials of a Bernoulli sequence with constant probability p. The author claims that the postulate $f_n \to p$ in von Mises' theory of collectives implies restrictions which contradict the notion of randomness. The criticism would equally apply to the law of large numbers or to the theorem that almost all numbers are normal. Actually the author only confuses the existence of a limit with the existence of a uniform limit. To remove the alleged difficulty the author modifies the definition of collectives essentially to the effect that it is only required that $|f_n-p| < \epsilon$ for a subsequence of n with density one. *W. Feller* (Ithaca, N. Y.).

Ottaviani, G. Sulle funzioni indipendenti. Giorn. Ist. Ital. Attuari **11**, 270–282 (1940). [MF 16619]

L'auteur rappelle l'essentiel de la "théorie abstraite" du calcul des probabilités (qui ramène l'étude des ensembles dénombrables de variables aléatoires dépendantes ou non à celle d'un ensemble convenablement choisi de fonctions mesurables-L d'une même variable), en soulignant le priorité de F. P. Cantelli dans l'édification de cette théorie, et envisage le problème précis suivant: t variant dans l'intervalle $(0, 1)$ et $f(t)$ étant une fonction mesurable sur $(0, 1)$, à quelles conditions doit satisfaire $f(t)$ pour qu'il existe au moins une fonction mesurable $g(t)$ indépendante de $f(t)$, c'est-à-dire telle que, quels que soient les ensembles de valeurs A et B, on ait

$$m\{E[f(t) \in A] \cap E[g(t) \in B]\} = m\{E[f(t) \in A]\} \cdot m\{E[g(t) \in B]\}.$$

Il démontre que, si $f(t)$ est croissante (ou décroissante) sur un intervalle (a, b) et si, sauf pour un ensemble de mesure nulle de valeurs de t, elle ne reprend pas en dehors de (a, b) de valeurs déjà prises sur (a, b), il n'existe pas de fonction $g(t)$ mesurable et indépendante de $f(t)$, en dehors de la solution triviale $g(t)=$constante; des résultats analogues et

plus généraux peuvent être obtenus. Inversement, l'auteur démontre que, si $f(t)$ ne prend qu'une infinité dénombrable de valeurs distinctes, on peut trouver $g(t)$ mesurable et indépendante de $f(t)$ et telle que, pour tout y, $m\{E[g(t)\leqq y]\} = G(y)$, où $G(y)$ est une fonction de répartition assignée d'avance. *R. Fortet* (Caen).

* **Jessen, Børge. A problem in geometrical probability.** Festskrift til Professor, Dr. Phil. J. F. Steffensen fra Kolleger og Elever paa hans 70 Aars Fødselsdag 28. Februar 1943, pp. 72–75. Den Danske Aktuarforening, Copenhagen, 1943. (Danish)

Consider in the plane n independent random vectors for which all directions are equally probable and whose lengths are given numbers $a_1, \cdots, a_n$. Let $P(r; a_1, \cdots, a_n)$ be the probability that their sum S_n does not exceed r in absolute value. The author first observes that this random walk problem is connected with Weyl's [Amer. J. Math. 60, 889–896 (1938)] theory of mean motion of the trigonometric polynomial $\sum_{k=1}^{n} a_k \exp\{i(\lambda_k t+\mu_k)\}$. The constant $c=\sum W_k\lambda_k$ of mean motion is determined by

$$W_k = P(a_k; a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n).$$

It follows that these probabilities satisfy the identity $\sum W_k = 1$. The author gives a simple direct proof for this relation based on induction and probability considerations. *W. Feller* (Ithaca, N. Y.).

Cantelli, F. P. Osservazioni sulla nota "Su una teoria astratta del calcolo delle probabilità e sulla sua applicazione al teorema detto 'delle probabilità zero e uno.'" Giorn. Ist. Ital. Attuari 11, 101–106 (1940). [MF 16625]

Let Π be a property of countable sequences of real numbers such that every sequence $r_1, r_2, \cdots, r_n, \cdots$ either has the property Π or the complementary property. A value r_n is called compatible with Π if there exists at least one sequence $r_1, r_2, \cdots, r_{n-1}, r_n, r_{n+1}, \cdots$ which has property Π. Let $f_1(t), f_2(t), \cdots, f_n(t), \cdots$ be a sequence of measurable functions in $(0, 1)$. We denote by E the subset of $(0, 1)$ such that, for $t\varepsilon E$, the sequence $f_1(t), f_2(t), \cdots$ has property Π. By E_n we denote the subset of $(0, 1)$ such that, for $t\varepsilon E_n$, $f_n(t)$ is compatible with Π. Clearly $E_n \supset E$ and thus $E_1E_2\cdots E_n \supset E$ for every n; hence $E_1E_2\cdots E_n\cdots = \lim_{n\to\infty} E_1E_2\cdots E_n = E+A$, with measure $m(A)$ positive or zero. The author proves (1) if the measure $m(E)$ is positive, then the ratio of measures $m(E)/m(E_1E_2\cdots E_n - A)$ has the limit 1 for $n\to\infty$. An immediate consequence is (2) if the property Π does not depend on the first n terms of a sequence for any value of n, then $m(E)$ is either zero or one.

These theorems have the following probabilistic interpretation. Let $\{X\} = X_1, X_2, \cdots, X_n, \cdots$ be a sequence of random variables, dependent or independent, and let $\{x\} = x_1, x_2, \cdots, x_n, \cdots$ denote a sequence of values assumed by those random variables. Let P be the probability that $\{x\}$ has property Π; let P_n be the probability that $x_1, \cdots, x_n$ are each compatible with Π and that either the sequence $\{x\}$ has property Π or that, for some $n_0 > n$, x_{n_0} is not compatible with Π. Then we have (1a) $\lim_{n\to\infty} P/P_n = 1$; (2a) if Π is a property which does not depend on the first n terms of a sequence for any value of n, then P is either zero or one (theorem of Jessen and Kolmogoroff). Theorem (1a) is similar but not equivalent to a theorem of P. Lévy [Théorie de l'Addition des Variables Aléatoires, Gauthier-Villars, Paris, 1937, p. 129] which also is a generalization of the theorem of Jessen and Kolmogoroff. *Z. W. Birnbaum.*

Hsu, P. L., et Chung, K. L. Sur un théorème de probabilités dénombrables. C. R. Acad. Sci. Paris 223, 467–469 (1946).

Let M be a set of numbers which form a group with respect to addition. Let $X_1, X_2, \cdots$ be a sequence of independent chance variables with the same distribution. Let $S_n = \sum_{i=1}^{n} X_i$. It is proved that the probability that $S_n \varepsilon M$ for infinitely many n is zero or one according as the series $\sum_{n=1}^{\infty} P\{S_n \varepsilon M\}$ converges or not. *J. Wolfowitz.*

Eyraud, Henri. Sur l'aléatoire fermée à une dimension. Ann. Univ. Lyon. Sect. A. (3) 4, 61–63 (1941).

Elementary properties of distributions on the perimeter of a circle. *J. L. Doob* (Urbana, Ill.).

Lourier, A. La loi directe, inverse et absolue des grands nombres. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 546–549 (1945).

Consider an infinite sequence of random variables η_n with distribution functions $F_n(y|\zeta)$, where ζ is a parameter to be considered as a random variable with distribution function $\theta(z)$. Under natural measurability conditions any functional $G(F_n)$ defines a random variable $\xi_n = f_n(\zeta)$ and the absolute distribution laws of both ξ_n and η_n can be computed in terms of F_n and θ. For a fixed $\epsilon > 0$ let A_n denote the event $|\eta_n - \xi_n| < \epsilon$. Let Pr (A_n) be the absolute probability of A_n, and denote by Pr $(A_n|z)$, Pr $(A_n|x_n)$, Pr $(A_n|y_n)$ the conditional probabilities of A_n under the hypotheses $\zeta = z$, $\xi_n = x_n$, $\eta_n = y_n$, respectively, where z, x_n, y_n are real numbers. The direct laws of large numbers assert that, under certain conditions, Pr $(A_n|z)\to 1$ for every fixed z. By an inverse law the author understands an assertion that Pr $(A_n|y_n)\to 1$; by an absolute law, that Pr $(A_n)\to 1$. Several theorems are proved of which the following are typical: Pr $(A_n|z)\to 1$ implies Pr $(A_n)\to 1$; conversely, the latter relation implies that for every $\delta > 0$ both Pr (Pr $(A_n^*|y_n) > \delta)\to 0$ and Pr (Pr $(A_n^*|z) > \delta)\to 0$, where A_n^* denotes the complementary event to A_n. *W. Feller* (Ithaca, N. Y.).

Núñez Bazalar, Tomás. On the law of large numbers of the theory of probability. Revista Ci., Lima 47, 601–643 (1945). (Spanish)

The author considers Bernoulli trials with $p = 3/5$. Replacing in Chebyshev's inequality the second moment by moments of order $3, \cdots, 6$ he obtains different rough estimates for the number of trials required in order that the probability that the frequency of successes differ from 3/5 by at most .02 is greater than .999. [Better estimates with less computational effort could be obtained by the customary use of Stirling's formula.] *W. Feller.*

Rosenblatt, Alfred. On the strong law of large numbers. Actas Acad. Ci. Lima 8, 7–26 (1945). (Spanish)

Let S_n be the number of successes in the first n trials of an infinite succession of Bernoulli trials. The author calculates an explicit estimate for an $\epsilon = \epsilon(n)$ such that the inequality $\Pr\{|S_n/n - p| < (1+\epsilon)((2pq\log\log n)/n)^{\frac{1}{2}}\} > 1-\epsilon$ will hold for all n. Similar estimates are obtained for the strong law of large numbers. *W. Feller* (Ithaca, N. Y.).

Ghosh, Manindra Nath. On the order of approximation involved in Laplace's central limit theorem in probability. Sankhyā 7, 323–326 (1946).

The Petrowski-Khintchine proof of the central limit theorem for random variables with the same distribution function $F(x)$ [Khintchine, Asymptotische Gesetze der

Wahrscheinlichkeitsrechnung, Ergebnisse der Math., v. 2, no. 4, Berlin, 1933, pp. 1 ff.] is retraced so as to yield an explicit estimate for the error term in the case where $F(x)$ has a moment of order $p>2$. This estimate is of the order $n^{-\alpha}$ with $\alpha=(p-2)/(11p-14)$. In the case $p=3$ it is well known that the actual magnitude of the error is $n^{-\frac{1}{2}}$.

W. Feller (Ithaca, N. Y.).

Feller, W. A limit theorem for random variables with infinite moments. Amer. J. Math. **68**, 257–262 (1946). [MF 16422]

Soit (X_k), $k=1, 2, \cdots$, une suite indéfinie de variables aléatoires indépendantes de même fonction de répartition $F(x)$, et (a_n), $n=1, 2, \cdots$, une suite de nombres certains positifs; en posant $S_n=\sum_{k=1}^{n}X_k$, soit L l'événement que l'inégalité $|S_n|>a_n$ a lieu pour une infinité de valeurs de n. L'auteur démontre, de façon élémentaire, que la probabilité de L est égale à 1 ou à 0 suivant que la série $\sum\int_{|x|\geq a_n}dF(x)$ diverge ou converge, lorsque les hypothèses suivantes sont réalisées: (a) $\int_{-\infty}^{+\infty}|x|dF(x)<\infty$, $\int_{-\infty}^{+\infty}xdF(x)=0$; il existe un nombre δ, $0<\delta<1$, tel que $\int_{-\infty}^{+\infty}|x|^{1+\delta}dF(x)=\infty$; (b) la suite a_n/n est décroissante, et il existe un nombre ϵ, $0\leqq\epsilon<1$, tel que la suite $a_n/n^{1/(1+\epsilon)}$ soit croissante. Ce résultat subsiste avec les hypothèses un peu différentes suivantes: (a′) $\int_{-\infty}^{+\infty}|x|dF(x)=\infty$; (b′) a_n/n est une suite croissante.

La méthode et les résultats peuvent s'étendre au cas où les X_k ont des fonctions de répartition $F_k(x)$ dépendant de k mais différant suffisamment peu d'une fonction de répartition fixe $F(x)$.

R. Fortet (Caen).

Kac, M. On the average of a certain Wiener functional and a related limit theorem in calculus of probability. Trans. Amer. Math. Soc. **59**, 401–414 (1946). [MF 16463]

Ce mémoire est à l'origine d'un travail par P. Erdős et l'auteur [Bull. Amer. Math. Soc. **52**, 292–302 (1946); ces Rev. **7**, 459]. On évalue ici la valeur moyenne sur l'espace de Wiener de la fonctionnelle

$$\exp\left\{-z\int_0^1|x(t)|dt\right\},\qquad z>0;$$

le résultat est équivalent à un théorème limite du calcul des probabilités [loc. cit., théorème IV]. (1) Comme dans l'ouvrage cité, on montre que l'on peut choisir des variables aléatoires particulières $X_1, X_2, \cdots$. (2) En résolvant une équation intégrale on obtient, sous forme explicite, la valeur moyenne de

$$\exp\{-(z/n^{\frac{1}{2}})(|X_1|+|X_1+X_2|+\cdots+|X_1+\cdots+X_n|)\}$$

en termes des fonctions de Bessel. (3) On passe à la limite $(n\to\infty)$, en établissant auparavant diverses propriétés (dont certaines sont nouvelles et présentent un intérêt par elles-mêmes) des fonctions de Bessel d'ordre élevé.

M. Loève (London).

Orts, J. M.ª A reciprocity property of the characteristic function. Revista Mat. Hisp.-Amer. (4) **6**, 43–47 (1946). (Spanish) [MF 16708]

Lévy, Paul. Intégrales stochastiques. Ann. Univ. Lyon. Sect. A. (3) **4**, 67–74 (1941).

General remarks on the "stochastic integral" $\int_a^b f(x)dg(x)$ obtained as the limit of the sum

$$\sum_{i=1}^{n}\{f(x_i^{(n)})+f(x_i^{(n)})\}\{g(x_i^{(n)})-g(x_{i-1}^{(n)})\},$$

where $x_0^{(n)}=a$, $x_n^{(n)}=b$, and $x_1^{(n)}, \cdots, x_{n-1}^{(n)}$ are obtained by choosing $X_1, \cdots, X_{n-1}$ independently in the interval (a, b), in accordance with a uniform probability distribution, and then ordering them. [Cf. Lévy, C. R. Acad. Sci. Paris **212**, 1066–1068 (1941); these Rev. **5**, 126.] If f, g provide the parametric representation of Hilbert's continuous curve which fills an isosceles right triangle of area s, the stochastic integral does not exist, but a slightly generalized one will exist, with value $s/2$.

J. L. Doob (Urbana, Ill.).

Hitosi-Iyoï. Calcul explicite de la distance de deux lois de probabilités. Ann. Univ. Lyon. Sect. A. (3) **3**, 55–63 (1 p. errata) (1940).

Let L, L' be two probability distributions. The author takes up the problem of calculating Lévy's distance $(d(L, L')=\inf d(X, X')$ for all X, X', where $d(X, X')$ is the distance according to any definition of distance between two chance variables with the given distributions and otherwise unrestricted) explicitly. He takes $d(X, X')^2$ as the expectation of $(X-X')^2$ and solves the problem for two distributions concentrated at two common points. He also treats the three point case, but according to the errata his treatment in that case is not complete.

J. L. Doob.

Pólya, George. Sur une généralisation d'un problème élémentaire classique, importante dans l'inspection des produits industriels. C. R. Acad. Sci. Paris **222**, 1422–1424 (1946).

The one-dimensional random walk problem with two absorbing barriers is reformulated by a slight change of notation in the following way. A moving point starts at the origin of the (x, y)-plane and moves in unit steps which are parallel either to the x-axis or the y-axis. The corresponding probabilities are p and $q=1-p$. The process stops when the point leaves the domain D defined by $-h_1+s(x+y)<x<h_2+s(x+y)$; here $h_1, h_2>0$ and $s/(s-1)=a/b$, where a and b are positive integers without common divisor. Clearly there are two "boundaries" F_p and F_q consisting of points (x, y) outside of D for which, respectively, $(x-1, y)$ or $(x, y-1)$ is within D. For any $(x, y)\epsilon D$ let $K(x, y)$ denote the number of possible paths from the origin to (x, y); for the points on the boundaries put $K(x, y)=K(x-1, y)$ or $K(x, y)=K(x, y-1)$. Clearly D is composed of subdomains D_k $(k=0, 1, \cdots)$ such that, when (x, y) runs through D_k, then $(x+a, y+b)$ runs through D_{k+1}. It follows that there exist integers $r, c_1, \cdots, c_r$ $(c_0=1)$ such that

$$c_rK(x, y)+c_{r-1}K(x+a, y+b)+c_{r-2}K(x+2a, y+2b)+\cdots+K(x+ra, y+rb)=0,$$

provided (x, y) is in D but not in D_0. Now the probability that the process ends at a point of F_p is $\sum K(x, y)p^xq^y$, the summation extending over all points of F_p. It follows, therefore, that both this probability and the expectation of the duration of the process are rational functions of p. These results can be applied to A. Wald's recent related investigations connected with sequential analysis [Ann. Math. Statistics **15**, 283–296 (1944); **16**, 117–186 (1945); these Rev. **6**, 88; **7**, 131]. Numerical applications are to follow.

W. Feller (Ithaca, N. Y.).

***Bogolyubov, N. O Nekotoryh Statističeskih Metodah v Matematičeskoĭ Fizike. [On Some Statistical Methods in Mathematical Physics].** Akademiya Nauk Ukrainskoĭ SSR, 1945. 139 pp. (Russian)

The first section of the book is devoted to the study of the asymptotic properties of the system of differential equa-

tions (1) $dx/dt=\epsilon X(t,x)$ as $\epsilon\to 0$; x, X represent vectors in n space. The theorems proved are of the following type: it is supposed that $\lim_{T\to\infty} T^{-1}\int_0^T X(t,x)dx=X_0(x)$ exists; it is concluded that, for sufficiently small ϵ, a solution $x(t)$ of (1) is arbitrarily close, uniformly in a prescribed finite interval, to the solution $\xi(t)$ of (1') $d\xi/dt=\epsilon X_0(x)$ for which $\xi(0)=x(0)$. It is then shown how, with the addition of further hypotheses on $X(t,x)$, the approximation can be made uniformly valid for all t. This makes it possible to conclude, for example, that, if $X(t,x)$ is periodic or almost periodic in t, there are solutions of (1) with these same properties.

The second section is devoted to the study of the effect of chance perturbations on a harmonic oscillator. The equation (2) $q''+\omega^2 q=f(t)$ is considered, in which $f(t)$, the perturbation term, is the sample function of a stationary Gaussian stochastic process. The distribution of the solution $q(t)$ is found and the pair q, q' is shown to determine a Markov process in the limiting case when the frequency spectrum of the $f(t)$ process has constant spectral density. [Cf. Doob, Ann. Math. Statistics **15**, 229–282 (1944); Blanc-Lapierre and Fortet, C. R. Acad. Sci. Paris **222**, 467–468 (1946); these Rev. **6**, 89; **7**, 460 for a treatment of this and related problems from a different point of view. Cf. also the author's exposition in Uchenye Zapiski Moskov. Gos. Univ. Fizika **77**, 51–73 (1945); these Rev. **7**, 314.] The perturbation term $f(t)$ is then replaced by $\epsilon f(t)$ and the asymptotic characteristics, as $\epsilon\to 0$, $t\to\infty$, of the distributions are found. Finally, the approach to equilibrium of a system coupled to a thermostat is analyzed in a particular case: a harmonic oscillator (position coordinate q) is coupled to a system in equilibrium (Gibbs canonical probabilities) consisting of a large number of harmonic oscillators (position coordinates $q_1, q_2, \cdots$). The mutual energy term is $\epsilon\sum_n \alpha_n q_n q$. The limiting distributions of q, q' are found when $\epsilon\to 0$, $t\to\infty$. *J. L. Doob* (Urbana, Ill.).

Sarymsakov, T. A. Sur les chaines de Bruns. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 241–243 (1945).

Consider an infinite sequence of independent trials with the possible results E_i and corresponding probabilities p_i, $i=1, \cdots, s$. Let k and r be fixed integers, $1\leqq k<r$. By definition, the mth term of a Bruns chain results in success E or failure F according as the subsequence of the original trials between $mk+1$ and $mk+r$ does or does not result in the succession $E_1, E_1, \cdots, E_1$. Accordingly, the Bruns chain is formed by dependent trials with a simple alternative. The author introduces instead of F the s^r-1 possible combinations of E_i as states and thus reduces the Bruns chain to a simple Markov chain. It follows, in particular, that the number of successes is, asymptotically, normally distributed. *W. Feller* (Ithaca, N. Y.).

***Blanc-Lapierre, André. Sur certaines fonctions aléatoires stationnaires. Application à l'étude des fluctuations dues à la structure électronique de l'électricité.** Publ. Labor. École Norm. Sup., Sér. Math., no. 1, 1945. 80 pp.

The author's thesis contains and amplifies his numerous papers which have appeared in C. R. Acad. Sci. Paris since 1943. In summary, the work is a theoretical study of the voltage fluctuations in the output of a linear amplifier subjected to impulses at random times $\{t_j\}$, $x(t)=\sum_j f(t-t_j)$; the t_j are supposed distributed in a Poisson distribution. The $x(t)$ stochastic process is a stationary process, which is approximately Gaussian if the rate ρ of impulses is high. The distributions are calculated in the general case, however, as functions of ρ. The necessary laws of large numbers are proved to show that probability means are also time means. The vacuum tube flicker effect is treated in detail. [For similar work from a different point of view see S. O. Rice, Bell System Tech. J. **23**, 282–332 (1944); **24**, 46–156 (1945); these Rev. **6**, 89, 233. Rice stresses the derivation of $x(t)$ as a sum of periodic terms with random coefficients as against the author's approach, in which the harmonic analysis is a by-product.] *J. L. Doob* (Urbana, Ill.).

Blanc-Lapierre, A. Effet Schottky. Fluctuations dans les amplificateurs linéaires et dans les détecteurs. Bull. Soc. Française Électriciens (6) **5**, 9 pp. (1945). [MF 16953]

Outline of some of the author's work reviewed above. *J. L. Doob* (Urbana, Ill.).

Blanc-Lapierre, André, et Fortet, Robert. Sur la structure des fonctions aléatoires strictement stationnaires à spectre totalement discontinu. C. R. Acad. Sci. Paris **222**, 1155–1157 (1946). [MF 16851]

Let $\omega_0, \omega_1, \omega_2, \cdots$ be a finite or infinite sequence of distinct real numbers and let $X_0, X_1, \cdots$ be a sequence of complex-valued chance variables which have zero means, are mutually uncorrelated, and the sum of whose second moments converges. The series $S(t)=\sum_n e^{i\omega_n t}X_n$ then defines a stochastic process with real parameter t, which is stationary in the wide sense, that is, in terms of the first two moments. The authors state that the following conditions must be satisfied if $S(t)$ is strictly stationary, that is, stationary in terms of distributions. (1) If there is a set of ω_j, say $\omega_1, \cdots, \omega_N$, for which there are no integers $\lambda_1, \cdots, \lambda_N$, not all zero, with (A) $\sum_1^N \lambda_j\omega_j=0$, then $\arg X_1, \cdots, \arg X_N$ are mutually independent and independent of the set $|X_1|, \cdots, |X_N|$, and are individually uniformly distributed in the interval $(0, 2\pi)$. (2) If there is a set of ω_j, say $\omega_1, \cdots, \omega_N$, for which there is a set of integers satisfying (A), with $\lambda_N\neq 0$, but for which there is no such relation between $N-1$ of the ω_j, then $\sum_1^N \lambda_n \arg X_n$ is independent of the set $(\arg X_1, \cdots, \arg X_{N-1})$ but may depend arbitrarily on the $|X_n|$. In particular, it is stated that the most general strictly stationary $S(t)$, if $\omega_0, \omega_1, \cdots$ is the sequence $0, \omega, 2\omega, \cdots$, is given by $\sum_0^\infty R_n e^{i(\psi_n+n\omega t+n\theta)}$, $R_n\geqq 0$, where θ is independent of the R_j, ψ_j taken together and is uniformly distributed in $(0, 2\pi)$. *J. L. Doob* (Urbana, Ill.).

Loève, Michel. Sur les fonctions aléatoires stationnaires de second ordre. Revue Sci. **83**, 297–303 (1945).

Expository article. *J. L. Doob* (Urbana, Ill.).

Brard, Roger. Interdépendance du tourbillon moyen local et de la vitesse moyenne locale d'agitation dans les mouvements turbulents. C. R. Acad. Sci. Paris **218**, 144–146 (1944). [MF 13464]

Brard, Roger. Sur la répartition du tourbillon dans un écoulement turbulent statistiquement permanent. C. R. Acad. Sci. Paris **219**, 604–605 (1944). [MF 15296]

In the first paper the author gives a justification of Gebelein's law [Gebelein, "Turbulenz," Berlin, 1935, pp. 76–84] that in a turbulent fluid the standard deviation of the instantaneous velocity is proportional to the $\frac{1}{4}$ power of the mean absolute value of the vorticity. According to the second paper, he now considers his justification unacceptable, but derives a partial result in the same direction, also obtained by Gebelein: the standard deviation of the vorticity is proportional to the square root of the mean of its absolute value. *J. L. Doob* (Urbana, Ill.).

Rivlin, R. S. An extension of Campbell's theorem of random fluctuations. Philos. Mag. (7) **36**, 688–693 (1945). [MF 16975]

Let $s(t)$ be such that $s(t)=0$ for $t<0$ and $s(t)$ vanishes with sufficient rapidity as $t\to\infty$. Let $y(t)=\sum_{-\infty}^{M}s(t-t_j)$, where $M, t_1, t_2, \cdots$ are independent random variables such that the t's are uniformly distributed in $(0, T)$ and M has the Poisson distribution with mean λT (λ a given constant). The author calculates the integral moments of $y-\bar{y}$ (or rather, the limits of these moments as $T\to\infty$). These results overlap with those of Rice [Bell System Tech. J. **23**, 282–332 (1944); these Rev. **6**, 89] but were obtained independently. [The reviewer would like to point out that one can easily calculate the moment generating function $\overline{\exp(zy(t))}$, which comes out to be equal to

$$\exp\left\{\lambda\int_0^T(\exp zs(t-\tau)-1)d\tau\right\} \tag{1}$$

and which yields the desired moments in a trivial fashion. A formula analogous to (1) with $z=i\xi$ has been used in a similar connection by M. Kac and H. Hurwitz, Ann. Math. Statistics **15**, 173–181 (1944), in particular, p. 177; these Rev. **6**, 89.] The author also calculates the moments of $y-\bar{y}$ when y is defined as

$$y(t)=\sum_{r=1}^{N}\sum_{j=0}^{M_r}s_r(t-t_{rj}),$$

where $M_1, \cdots, M_N, t_{11}, t_{12}, \cdots$ are again independent random variables, each t being uniformly distributed in $(0, T)$ and M_r having Poisson's distribution with mean $\lambda_r T$. Again it is simpler to calculate the moment generating function, which in this case comes out to be

$$\prod_{r=1}^{N}\exp\left\{\lambda_r\int^T(\exp zs_r(t-\tau)-1)d\tau\right\}.$$

M. Kac (Ithaca, N. Y.).

Boonimovich, V. Effect of the fluctuations and signal voltages on a non-linear system. Acad. Sci. USSR. J. Phys. **10**, 35–48 (1946).

The spectrum of signal plus noise output of a nonlinear device is determined. As the author points out himself (in a note added in proof), the method is essentially that used by Van Vleck and North and discussed by Rice in the second part of his monograph [Bell System Tech. J. **24**, 46–156 (1945); these Rev. **6**, 233]. *M. Kac* (Ithaca, N. Y.).

Berlovich, E. Statistics of misses in Geiger-Müller counters. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **16**, 543–546 (1946). (Russian. English summary)

The author criticizes the accepted theory of misses in Geiger-Müller counters and develops a new formula. It seems to the reviewer that the criticism is not justified and that the new formula is erroneous (a probability is computed by integration over cases which are not mutually exclusive). *W. Feller* (Ithaca, N. Y.).

Berlovich, E. The theory of misses in an electromagnetic numerator at the output of a dividing scheme with a Geiger-Müller counter. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **16**, 547–552 (1946). (Russian. English summary)

A particular circuit for the numerator is studied. The formula derived for the misses is based on the author's result in the paper reviewed above. *W. Feller.*

Mann, H. B. A note on the correction of Geiger Müller counter data. Quart. Appl. Math. **4**, 307–309 (1946).

An improved estimate of the error term in the formula given by Kurbatov and Mann [Phys. Rev. (2) **68**, 40–43 (1945); these Rev. **7**, 18]. *W. Feller* (Ithaca, N. Y.).

Mathematical Statistics

***Cramér, Harald. Mathematical Methods of Statistics.** Princeton Mathematical Series, vol. 9. Princeton University Press, Princeton, N. J., 1946. xvi+575 pp. $6.00.

The present book is, remarkably enough, both a treatise and a textbook. The presentation is such that the theory of probability is the all important background, whereas statistical techniques and theories emerge as applications. In fact, mathematical statistics is presented as a part of general probability theory. However, in presenting the fundamental ideas and results of modern statistics the author never loses sight of the motivation behind them. He does not choose the easy way of presenting statistics as a sort of an appendix to probability theory (had he done that the result might have looked somewhat like treating mechanics as an appendix to the theory of differential equations) but he makes it stand up as the "applied" counterpart of the "pure" theory. Thus while making it clear that the mathematical methods and techniques of statistics are but a part of the mathematical apparatus of probability theory, he emphasizes with equal clarity that the problems and motivations of statistics are peculiarly its own.

The book is divided into three parts: mathematical introduction; random variables and probability distributions; statistical inference. The first part contains a rather extensive presentation of the theory of measure and integration [85 pages], a short [14 pages] but lucid chapter on characteristic functions, a self-contained exposition of the theory of matrices [19 pages] and finally "miscellaneous complements," which include such topics as the gamma and beta functions, Stirling's formula and orthogonal polynomials.

The second part provides the bridge between probability theory and statistics. It treats the fundamental topics of random variables, independence, the central limit theorem (in one and more dimensions), the law ("weak") of large numbers and in addition includes a discussion of the more common distributions like the binomial, Poisson, rectangular, normal (in one and more dimensions), χ^2, t, etc. The normal distribution is discussed with particular thoroughness in view of its fundamental importance in statistics. There is a short but adequate discussion of Gram-Charlier and Edgworth series. In proving various theorems (for example, the multidimensional central limit theorem) the author does not strive to obtain ultimate generality. He prefers to prove them under conditions most often met in practice and to refer the interested reader to original memoirs on the subject.

The third and largest [253 pages] part of the book is devoted to modern statistical theories. The ideas of sampling and sampling distributions are introduced with great clarity. Tests of significance, general theory of testing statistical hypotheses and theories of estimation are treated systematically and rigorously. There is an excellent discussion of such classical topics as goodness of fit, Sheppard's correction, unbiased estimates, method of maximum likeli-

hood, etc. A chapter on analysis of variance, though short [12 pages], is quite satisfactory. The same can be said about the last chapter of the book, which deals with regression problems.

Numerous well chosen and interesting exercises and problems add greatly to the value of the book. Historical notes and references to the literature are also quite helpful. The book is remarkably free of misprints. Tables of the normal distribution and percentage points of the normal, χ^2 and t distributions are included.

The reviewer feels that the amount of space devoted to the theory of measure and integration (about one sixth of the book) is somewhat out of balance with the rest of the book. The importance of the theory of measure as a basis for probability theory could be appreciated more fully if "strong" theorems like the strong law of large numbers or the law of the iterated logarithm were considered. The book leaves these topics out, however. For the actual purposes of the book a much more rudimentary survey of measure and integration would prove quite sufficient. On the other hand, by including this extensive and readable account of the theories of measure and integration the author makes them easily available and this will undoubtedly prove beneficial. The reviewer also feels that the author's approach to the notion of a random variable through a system of axioms is somewhat unnatural. It might have been simpler to define random variables as measurable functions on spaces on which a Lebesgue measure has been established (the measure of the whole space being 1). With this approach "convergence in probability" would become "convergence in measure" (sometimes called asymptotic convergence) and "convergence with probability 1" (not treated in this book) convergence almost everywhere. The above criticism is, however, not very serious and it does not detract from the great value of the work. The reviewer is certain that this book will become a standard reference and a standard text for many years to come. *M. Kac* (Ithaca, N. Y.).

Austen, A. E. W., and Pelzer, H. Linear 'curves of best fit.' Nature **157**, 693–694 (1946). [MF 16836]

It is assumed that there is some exact relation $w_1=\rho v_1$ relating two variables w_1 and v_1 whose observed values, w and v, are subject to error. When both w and v contain errors the authors estimate ρ by minimizing the sum of values of $\Delta x^2+\Delta y^2$, where $x=v$, $y=\beta w$, $\beta=\sigma_v/\sigma_w$; here v, w are measured from their means and Δx, Δy are the deviations of a measured point from a point on a line representing the true relation.

The reviewer finds that the authors' estimate of ρ minimizes the sum of squares of the distances from the x, y points to a line and that it is algebraically equivalent to the well-known estimate $\rho=\beta^{-1}\tan\theta$, where

$$\tan 2\theta=2\sum xy/(\sum x^2-\sum y^2).$$

T. E. Sterne (Cambridge, Mass.).

***Busk, Thøger. Some remarks on the computation of the mean error for values graduated by the method of least squares.** Festskrift til Professor, Dr. Phil. J. F. Steffensen fra Kolleger og Elever paa hans 70 Aars Fødselsdag 28. Februar 1943, pp. 40–44. Den Danske Aktuarforening, Copenhagen, 1943. (Danish)

Let X and Y be two random variables obtained by graduation from a set of mutually independent variables (for example, the computed lengths of two sides of a triangle when all sides and angles were observed). The author computes the variance of linear combinations K_1X+K_2Y both for conventional least squares and for Thiele's method of free functions. *W. Feller* (Ithaca, N. Y.).

Vatnsdal, J. R. Minimal variance and its relation to efficient moment tests. Ann. Math. Statistics **17**, 198–207 (1946).

It is assumed that a curve is fitted by the method of moments and that the hypothesis that the population distribution is of the given form, with the parameters as estimated, is tested by examining the deviations of higher moments from their expected values. The author defines as the "most efficient" tests those in which each of the higher moments is computed about such a point that its variance is minimized. It is shown that such a point exists for each moment. Conditions on this point for the second and third moments are given. For symmetrical populations, this point is the mean. Several other such results are obtained, among them one for an asymmetrical Pearson distribution. Several examples are given. *J. Wolfowitz* (New York, N. Y.).

Gleissberg, W. Ein Kriterium für die Realität zyklischer Variationen. Rev. Fac. Sci. Univ. Istanbul (A) **10**, 36–42 (1945). (German. Turkish summary) [MF 16716]

The probability $W(n, k)$ that a series of n different random numbers should contain more than k extremal values has been tabulated by the author [same Rev. **10**, 25–35 (1945); these Rev. **7**, 406] and is used here as a test of significance of the cycles in an observational time series. The test is applied, with answer in the affirmative, to the $k=4$ cycles appearing in the series formed by the $n=15$ graduated maxima of 15 complete periods of sunspot numbers. [It has escaped the author's attention that his test is closely related to known results in the theory of monotonic "runs" of "phases" of a time series; thus his recursive formula for $W(n, k)$ can easily be obtained by summing over r from $k+2$ to $n-1$ in André's formula for the probability $p(n, r)$ that n observations should form precisely r runs; cf. W. A. Wallis and G. H. Moore, Nat. Bureau Econ. Res., Tech. Paper no. 1, New York, 1941; these Rev. **3**, 176.] *H. Wold* (Uppsala).

Mayot, Marcel. Sur la détermination statistique des composantes cycliques d'un phénomène: application aux étoiles variables. C. R. Acad. Sci. Paris **223**, 125–127 (1946).

Radhakrishna Rao, C. On the mean conserving property. Proc. Indian Acad. Sci., Sect. A. **23**, 165–173 (1946). [MF 16781]

Let the variables $x_1, \cdots, x_n$ have the elementary distribution law $\phi(x_i, \lambda_i, \mu_i, \cdots)$, $i=1, \cdots, n$. The probability density ϕ is said to have the mean conserving property $M^{\alpha\beta\gamma}_{\xi\eta\zeta}$ if, by forming the mean of n variables with distribution laws ϕ and fixed parameters $\alpha, \beta, \gamma, \cdots$ and with $\xi, \eta, \zeta, \cdots$ varying in any manner, we obtain again a variable having a distribution law $\phi(x, \lambda, \mu, \cdots)$ with some values of the parameters $\lambda, \mu, \cdots$ depending on $\lambda_i, \mu_i, \cdots$, $i=1, \cdots, n$. The author gives various necessary and sufficient criteria for a distribution function to have the mean conserving property. The semi-invariant generating function of a distribution law with the mean conserving property satisfies a homogeneous differential equation. The author proves the mean preserving property for several important types of distribution functions, as, for instance, distribution functions of Bessel function and gamma function type. *H. B. Mann* (Columbus, Ohio).

Radhakrishna Rao, C. On the linear combination of observations and the general theory of least squares. Sankhyā 7, 237–256 (1946).

In previous papers [Sankhyā 7, 9–16, 16–19 (1945); these Rev. 7, 132] the author generalized well-known results [for example, David and Neyman, Statistical Research Memoirs 2, 105–116 (1938); Kołodziejczyk, Biometrika 27, 161–190 (1935)] on the relation of the method of least squares to (1) unbiased linear estimates of minimum variance and (2) tests of linear hypotheses. The generalization lay in the removal of all restrictions on the (known) matrix of coefficients which determines the expected values of the observations as linear combinations of the (unknown) parameters. The concepts, methods, and results of these papers are applied in the present paper to various general problems of analysis of variance and covariance. The paper, which includes twenty theorems on point estimation, properties of the normal equations, distribution theory and statistical tests, does not lend itself to a more detailed summary here. [Note: the "method of sweep out" occurring in section 4 is described after equation (5.2), p. 12, of the earlier papers.]

H. Scheffé (Berkeley, Calif.).

Carlton, A. George. Estimating the parameters of a rectangular distribution. Ann. Math. Statistics 17, 355–358 (1946).

Wherry, Robert J., and Taylor, Erwin K. The relation of multiserial eta to other measures of correlation. Psychometrika 11, 155–161 (1946).

Hayes, Samuel P., Jr. Diagrams for computing tetrachoric correlation coefficients from percentage differences. Psychometrika 11, 163–172 (1946).

Guttman, Louis. The test-retest reliability of qualitative data. Psychometrika 11, 81–95 (1946). [MF 16898]

Suppose that a person is given infinitely many independent trials on an item to which his responses may fall into any one of m categories. The probability of a category for that person is the proportion of times his response falls into that category. The highest probability for any category is the modal probability; for the ith person let this be P_i. Then the reliability coefficient of the item for that person is defined as $R_i=m(m-1)^{-1}(P_i-1/m)$, $0\leqq R_i\leqq 1$. If the mean of P_i over all persons composing a population is denoted by α, then the reliability coefficient of the item for the population is defined to be $\rho=m(m-1)^{-1}(\alpha-1/m)$. Zero group reliability implies zero individual reliabilities; unit group reliability implies unit individual reliabilities. To obtain direct information about the R_i's would require actual repeated experiments under the same conditions but it is shown that it is possible to obtain an unbiased estimate of a lower bound for ρ by a single trial over a group of persons provided that it is assumed that in repeated trials the variation in responses from each person would be independent of the variation of every other person. It is furthermore shown that this estimate can be improved upon by means of another trial on a statistically independent item using the joint occurrences. Moreover, by the use of such a pair of trials two unbiased estimates of upper bounds for ρ can be found, one better than the other but slightly more complicated. The sampling problems which arise are recognized but not discussed in this paper. *C. C. Craig.*

Guttman, Louis. An approach for quantifying paired comparisons and rank order. Ann. Math. Statistics 17, 144–163 (1946).

Let a group of n individuals each be asked to assign ranks to n comparable objects. It is assumed that in doing this each person makes all the $n(n-1)/2$ comparisons between pairs and for the sequel no differentiation is made between assigning ranks and the use of paired comparisons for the problem of quantification here considered. This problem is to assign numerical values to each object which will best agree with individual judgments of the order among the objects. The principle adopted is to choose these numerical values so that for objects ranked higher or lower in comparisons the variation, as in the analysis of variance, will be a minimum within individuals as compared with the like variation for the group as a whole. The minimization of the variation is effected by the least squares principle. This method in general is shown not to be equivalent to estimating differences between means of hypothetical variables attached to the objects, with an assumed form of distribution of these variables; in fact, this approach is quite independent of such older methods and has the further feature of being applicable to the case of complex comparisons in which the objects may be combinations of items. This investigation arose from the effort to evolve an Army discharge point scheme which would best accord with the judgments of the soldiers themselves which involved complex comparisons. The problem in both cases, for ordinary and complex comparisons, is given a matrix formulation and the solution in each case appears as a vector associated with the largest root of a matrix obtained from the comparisons or rankings. Known iterative methods then can be used to get solutions. For the complex case some of the objects may be divided into quantitative categories in which case it may be desirable to make the numerical values linear functions of those quantitative values. This case is handled as well as the one in which no such restriction is imposed.

C. C. Craig (Ann Arbor, Mich.).

Kendall, M. G. The treatment of ties in ranking problems. Biometrika 33, 239–251 (1945). [MF 16457]

The Spearman coefficient of rank correlation ρ and the alternative coefficient τ defined by the author may be extended to cover the case of tied ranks, where each tied element in a group is assigned the average rank of the ranks covered by the group, in two different ways, denoted by S and W. If X, Y are the two rankings, $\rho_S=\rho(X, Y)$, while ρ_W is the covariance $(X, Y)/\frac{1}{12}(n^2-1)$ and the difference between τ_S and τ_W is similar. Then ρ_W (τ_W) may be considered as the average ρ (τ) which would be obtained if ties were replaced by appropriate integral rankings in all possible ways. It is argued that ρ_W, τ_W are preferable when there is an objective order, while ρ_S, τ_S are preferable if only concordance between two estimates is being measured. For m rankings, the method of ranking according to sums of ranks continues to yield the maximum average ρ_W between the estimated ranking and the individual rankings when there are ties in the individual rankings, unless the estimated ranking itself contains ties. The modifications of the usual test of significance of the coefficient of concordance appropriate for the case of tied rankings are discussed.

D. Blackwell (Washington, D. C.).

Tsao, Fei. General solution of the analysis of variance and covariance in the case of unequal or disproportionate numbers of observations in the subclasses. Psychometrika **11**, 107–128 (1946). [MF 16900]

The author considers the analysis of variance and covariance in certain tables with unequal frequencies in the various cells. Estimates of variance are presented under two linear hypotheses. These estimates require the solution of normal equations. Approximations are proposed without proof. *J. Wolfowitz* (New York, N. Y.).

Lindblom, Sven G. On the connection between tests of significance for correlation coefficients and for differences between means. Skand. Aktuarietidskr. **29**, 12–29 (1946).

From a small amount of basic distribution theory for certain functions of a set of independent standard normal deviates, the author derives the distributions of statistics used for testing whether observed correlation coefficients are "significantly different" from zero and whether certain means "differ significantly." The appropriate hypotheses being tested are not formulated as relations among population parameters. Generalizations are made to the multivariate case. [In equation (10) the denominator should be replaced by its square root.] *H. Scheffé.*

Hartley, H. O., Pearson, E. S., Thompson, Catherine M., and Merrington, Maxine. Tables for testing the homogeneity of a set of estimated variances. Biometrika **33**, 296–304 (1946). [MF 16865]

The problem of testing a set of mean squares, assumed to be obtained from normal populations, to ascertain whether the population variances are all equal was first explicitly attacked by Neyman and Pearson [Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. **1931**, 460–481] with the L_4 test. Bartlett [Proc. Roy. Soc. London. Ser. A. **160**, 268–282 (1937)] suggested a modified test, for which Hartley [Biometrika **31**, 249–255 (1940); these Rev. **1**, 346] provided an approximation formula. The present tables present and explain an adequate set of 5% and 1% points for samples from 3 to 15. They are given for two cases for $C_1=0.0(0.5)5.0(1)10(2)14$, where $C_1=\sum\nu_i^{-1}-(\sum\nu_i)^{-1}$ and ν_i is the number of degrees of freedom in the ith mean square. Linear interpolation and the use of the approximation combined are stated not to disturb two-decimal accuracy in any case when each $\nu_i\geqq 4$. *J. W. Tukey* (Princeton, N. J.).

Pearson, E. S. The probability integral of the mean deviation. Biometrika **33**, 252–253 (1945). [MF 16458]

This introduces the following three papers and contains unpublished formulae by R. C. Geary for the third and fourth semi-invariants of the mean deviation for samples from a normal population. *A. M. Mood* (Ames, Iowa).

Godwin, H. J. On the distribution of the estimate of mean deviation obtained from samples from a normal population. Biometrika **33**, 254–256 (1945). [MF 16459]

The distribution is derived of the mean deviation $m=n^{-1}\sum|x_i-\bar{x}|$ for samples of size n from a normal population with zero mean and unit variance. The density function is

$$f_n(m)dm=\frac{n^{\frac{1}{2}}}{2^{\frac{1}{2}(n+1)}\pi^{\frac{1}{2}(n-1)}}\times\left\{\sum_{k=1}^{n-1}\binom{n}{k}\exp\left(-\frac{m^2n^2}{8k(n-k)}\right)G_{k-1}(\tfrac{1}{2}nm)G_{n-k-1}(\tfrac{1}{2}nm)\right\}dm,$$

where

$$G_r(x)=\int_0^x\exp\left(-\frac{t^2}{2r(r+1)}\right)G_{r-1}(t)dt,\quad G_0(x)=1.$$

A. M. Mood (Ames, Iowa).

Hartley, H. O. Note on the calculation of the distribution of the estimate of mean deviation in normal samples. Biometrika **33**, 257–258 (1945). [MF 16460]

Tables of the probability integral of the mean deviation in normal samples. Biometrika **33**, 259–265 (1945). [MF 16461]

The cumulative distribution $\int_0^m f_n(t)dt$ of m is tabulated (five places) at intervals of .01 for $n=2, 3, \cdots, 10$.

A. M. Mood (Ames, Iowa).

Baldwin, Elizabeth M. Table of percentage points of the t-distribution. Biometrika **33**, 362 (1946). [MF 16869]

A table of the 5% and 1% points of Student's t-distribution is given for the number of degrees of freedom n taking values from 1 to 100. This extension of previously available tables beyond $n=30$ was undertaken since "it was found that the use of the normal probability curve, taking $t\{(n-2)/n\}^{\frac{1}{2}}$ as a normal deviate, gave results much smaller than the true values." *Z. W. Birnbaum* (Seattle, Wash.).

***Hald, A., and Rasch, G. Some applications of methods of transformation in the normal distribution theory.** Festskrift til Professor, Dr. Phil. J. F. Steffensen fra Kolleger og Elever paa hans 70 Aars Fødselsdag 28. Februar 1943, pp. 52–65. Den Danske Aktuarforening, Copenhagen, 1943. (Danish)

The authors apply the special transformation used by Steffensen [Skand. Aktuarietidskr. **19**, 108–125 (1936)] and W. Behrens [Landwirthschaftliche Jahrbücher **68**, 807 ff. (1929)] to derive neatly the fundamental distributions for Student-Fisher t-tests and variance-ratio tests in a number of the now standard situations. *C. C. Craig.*

Goldberg, Henry, and Levine, Harriet. Approximate formulas for the percentage points and normalization of t and χ^2. Ann. Math. Statistics **17**, 216–225 (1946).

Application of the Cornish-Fisher [Rev. Inst. Int. Stat. **4**, 1–14 (1937)] method to the normalization of t and χ^2 carrying the computations one term further in each case than Peiser [same Ann. **14**, 56–62 (1943); these Rev. **4**, 222]. Tables of exact and approximate percentage points and short tables of exact and approximate percentages at fixed points. The approximate χ^2 percentage points include other well-known approximations and extend those of Merrington [Biometrika **32**, 200–202 (1941); these Rev. **3**, 175] to degrees of freedom as small as unity.

J. W. Tukey (Princeton, N. J.).

Jones, A. E. A useful method for the routine estimation of dispersion from large samples. Biometrika **33**, 274–282 (1946). [MF 16863]

The possibility of estimating dispersions by using only the extreme observations in a sample is investigated. Let S be the difference between the sums of the r largest and r smallest values observed in the sample. Complicated expressions for the mean and variance of S are simplified by means of approximations but still remain rather complex. Further approximations are derived which assume the density function decreases exponentially on its tails. Tables are given by means of which the standard deviation of a

normal population may be estimated from S and the efficiency of the estimate is studied. The validity of this method of estimation probably depends strongly on the assumption of normality. *A. M. Mood* (Ames, Iowa).

Winsten, C. B. Inequalities in terms of mean range. Biometrika 33, 283–295 (1946). [MF 16864]

Let w_n denote the expected value of the range of a sample of size n. For a given population distribution d and for a given value t, let $p_d(x, t)$ denote the probability that a single observation falls inside the interval $(x, x+tw_n)$ and let $p_d(t)$ be the least upper bound of $p_d(x, t)$ with respect to x. The author derives a function $p(t)$ such that $p(t) \leqq p_d(t)$ holds for all t and all distributions d. It is shown that this inequality cannot be improved if nothing is known about the distribution d except the value of w_n for a given n. A similar inequality is derived for unimodal symmetrical distributions and this inequality is shown to be the best possible one of its type. To facilitate the use of the inequalities, diagrams are given for small values of n. Simple approximate formulas are given for large values of t. A similar method is applied to prove the Gauss-Winkler inequalities.

A. Wald (New York, N. Y.).

Baker, G. A. Distribution of the ratio of sample range to sample standard deviation for normal and combinations of normal distributions. Ann. Math. Statistics 17, 366–369 (1946).

Walsh, John E. On the power function of the sign test for slippage of means. Ann. Math. Statistics 17, 358–362 (1946).

Walsh, John E. Some order statistic distributions for samples of size four. Ann. Math. Statistics 17, 246–248 (1946).

Let y_i be the ith largest of a sample of four from an $N(0, 1)$ population. The author presents the joint distribution of the three statistics

$$y_4+y_3-y_2-y_1; \quad y_4-y_3+y_2-y_1; \quad y_4-y_3-y_2+y_1,$$

plus the distribution of each singly. [The notation is that of the reviewer.] *R. L. Anderson* (Raleigh, N. C.).

Ferris, Charles D., Grubbs, Frank E., and Weaver, Chalmers L. Operating characteristics for the common statistical tests of significance. Ann. Math. Statistics 17, 178–197 (1946).

Operating characteristics (type II errors) are plotted for the following tests, based on normal distribution theory. (1) One-sided tests of a variance using the χ^2 test: H_0 is $\sigma_1=\sigma$; H_1 is $\sigma_1=\lambda\sigma$, where H_0 denotes the null hypothesis and H_1 the set of alternatives; λ may have the range $\lambda>1$ or $0<\lambda<1$. (2) One-sided tests of the ratio of two variances using the F test: H_0 is $\sigma_1=\sigma_2$; H_1 is $\sigma_1=\lambda\sigma_2$, $\lambda>1$. This power function may also be used for the analysis of variance test if the true group means are a sample from a normal population. (3) Two-sided tests of the mean when the variance is known: H_0 is $\mu_1=\mu$; H_1 is $|\mu_1-\mu|=\lambda\sigma_1$, $\lambda>0$. The curves may also be used for the test of the difference between two means (both variances known). (4) Two-sided tests of a mean using the t test: H_0 is $\mu_1=\mu$; H_1 is $|\mu_1-\mu|=\lambda\sigma_1$, $\lambda>0$. The curves may also be used for the test of the difference of two means when the variances are equal. In all cases the type I error is taken to be .05 and the type II error is plotted against λ for selected sample sizes in the range 2 to 100. The curves may be read to only one or two significant figures, but this is sufficient accuracy for many practical purposes. *A. M. Mood* (Ames, Iowa).

Ville, J. Sur la transitivité d'une methode d'estimation. Ann. Univ. Lyon. Sect. A. (3) 7, 14–20 (1944).

The author gives a criterion for choosing which of two probability densities gave rise to a given sample value: complementary sets are chosen in a natural way and the probability density accepted depends on which of these two sets contains the sample value. In spite of the reasonableness of this procedure, it is shown that three densities f_1, f_2, and f_3 can be found such that on the basis of some sample values f_1 would be selected as the true density rather than f_2, f_2 rather than f_3 and f_3 rather than f_1. This disturbing lack of transitivity does not occur if the probability densities considered are restricted to lie in certain families, for example, if all the densities considered are of the form $f(x-\theta)$, where f is even and decreases when $|x|$ increases.

J. L. Doob (Urbana, Ill.).

Cochran, W. G. Relative accuracy of systematic and stratified random samples for a certain class of populations. Ann. Math. Statistics 17, 164–177 (1946).

The author establishes, through references to several contributions to statistical practice, that not infrequently examples prove a population under investigation to be such that the variance within a sample increases steadily as the size of the sample increases. Such must indeed occur if the elements are serially related, the correlation between two elements being a positive monotone decreasing function of the distance between the elements. For such a population, the author compares the relative efficiencies of (i) a systematic sample of every kth element, (ii) a stratified random sample with one element per sample and (iii) a random sample. Among other conclusions, he proves the following: no general result is valid for the relative efficiency of the systematic sample (as evidenced by crucial examples). If, however, the correlogram is "concave upwards" (some numerical examples are given for the linear and for the exponential case previously adopted by other workers) then the systematic sample is on the average more accurate than the stratified sample for any size of sample. In contrast to this negative result, the stratified random sample is always at least as accurate on the average as the random sample and its relative efficiency is a monotone increasing function of the size of the sample. Explicit formulas for the several variances are given throughout. Extensive references to recent work touching upon this problem are given.

A. A. Bennett (Providence, R. I.).

Festinger, Leon. The significance of difference between means without reference to the frequency distribution function. Psychometrika 11, 97–105 (1946). [MF 16899]

Given two samples, of m and $n(\leqq m)$, respectively, it is possible to test whether they come from the same population by combining them, ranking the combined sample, and calculating the absolute value of the mean ranks of the two original samples. The author has computed the distribution of the resulting statistic in detail for a range of values of m and n. He presents tables of the 1% and 5% points for $m+n=4(1)30$, $n=2(1)15$ and for $m+n=31(1)40$, $n=2(1)12$.

J. W. Tukey (Princeton, N. J.).

Camp, Burton H. The effect on a distribution function of small changes in the population function. Ann. Math. Statistics 17, 226–231 (1946).

This paper presents three theorems and corollaries establishing, for an arbitrary statistic, inequalities of the (δ, ϵ) type which serve to justify under very general conditions the customary uncritical acceptance of the notion that, if the actual population function is not very different from one assumed in theory, then the true sampling distribution of the statistic will not differ greatly from that obtainable from the theory. *A. A. Bennett* (Providence, R. I.).

Mathen, K. K. A criterion for testing whether two samples have come from the same population without assuming the nature of the population. Sankhyā 7, 329 (1946).

Let $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ be independent observations from populations with density functions $f(x)$ and $g(x)$, respectively, both of unknown functional form. It is desired to test the null hypothesis $f(x) \equiv g(x)$ against (presumably) alternatives of the form $f(x+d) \equiv g(x)$, $d>0$. The statistic s proposed is as follows. Let t_i be the number of y's less than x_i. Then $s=\sum_{i=1}^{m} t_i$. Small values of s are significant. The distribution of s under the null hypothesis is easy to obtain. A table of the distribution is being prepared.

J. Wolfowitz (New York, N. Y.).

Girshick, M. A. Contributions to the theory of sequential analysis. I. Ann. Math. Statistics 17, 123–143 (1946).

Let π_1 and π_2 be two populations with density functions $f(x, \theta_1)$ and $f(x, \theta_2)$, respectively. Both θ_1 and θ_2 are unknown and it is desired to decide sequentially between $H_0: \theta_1 < \theta_2$ and $H_1: \theta_1 > \theta_2$. Let (θ_1^0, θ_2^0) be a couple such that the risks of wrong decisions when (θ_1, θ_2) is (θ_1^0, θ_2^0) and when (θ_1, θ_2) is (θ_2^0, θ_1^0) are specified. Let (x_{1j}, x_{2j}) be the jth couple of observations on π_1, π_2. It is proposed to use the Wald sequential test by cumulating

$$\log \left\{ \frac{f(x_{2j}, \theta_1^0) f(x_{1j}, \theta_2^0)}{f(x_{2j}, \theta_2^0) f(x_{1j}, \theta_1^0)} \right\}, \qquad j=1, 2, \cdots.$$

It is proved that, if $E(\log f(x, \theta_2^0)/f(x, \theta_1^0))$ is a monotonic function of θ, the equi-power contours in the θ_1, θ_2 plane never cross the line $\theta_1=\theta_2$; in this case, therefore, the test is unbiased. It is shown that, when $f(x, \theta)$ admits a sufficient estimate of θ, the decision and power functions take specially simple forms. Application is made to special problems.

J. Wolfowitz (New York, N. Y.).

Plackett, R. L., and Burman, J. P. The design of optimum multifactorial experiments. Biometrika 33, 305–325 (1946). [MF 16866]

The authors consider factorial designs where each factor appears at the same number of values and the variances of the least square estimates of certain functions of the main effects are equal. Among such designs they find those whose common variances are at a minimum. Necessary and sufficient conditions for a design to have this property are: (1) each component is replicated at each of its values the same number of times; (2) each pair of components occurs together at every combination of values the same number of times; (3) the number of assemblies is divisible by the square of the number of values. Designs which have this property are termed multifactorial designs. Let N be the number of assemblies, L the number of values. For $L=2$ the authors construct these designs for all $N=2^h(p^n+1)=4K$ where p is either an odd prime or zero. This includes all numbers below 100 which are of the form $4K$ except the number 92. The authors also construct designs for $L=p$, $N=p^n$, where p is a prime, and discuss the relation between multifactorial designs and balanced incomplete block designs. *H. B. Mann* (Columbus, Ohio).

Plackett, R. L. Some generalizations in the multifactorial design. Biometrika 33, 328–332 (1946). [MF 16868]

Given a linearly independent set of not necessarily orthogonal linear functions of the main effects satisfying certain conditions, the author develops methods for constructing designs to estimate these functions with maximum precision. Also, if a multifactorial design is incomplete due to loss of observations, linear functions of the main effects, which can be estimated with maximum precision, can be constructed by the author's methods. *H. B. Mann.*

Lindley, D. V. On the solution of some equations in least squares. Biometrika 33, 326–327 (1946). [MF 16867]

The author discusses the analysis of multifactorial designs for the case $L=2$, when it is not possible to keep the factors exactly at the prescribed value. *H. B. Mann.*

Bliss, C. I. An experimental design for slope-ratio assays. Ann. Math. Statistics 17, 232–237 (1946).

In certain types of biological assay where a number of agents (for example, vitamins, poisons) are being compared, the expected response y_i to a dose x_i of the ith agent is $(\alpha'+\beta x_i)$, where α', the response to zero dose, is the same for all agents. The potency of the agent (i) relative to a standard agent (1) is then appropriately defined as β_i/β_1. The least squares estimates of α' and the β_i are worked out for an experiment in which (a) the series of doses of any agent is proportional to the corresponding series of doses of the standard, (b) there are r replicates of each dose of each agent, h replicates of each dose of the standard and h' replicates of the zero dose. Confidence limits for the relative potency β_i/β_1 are established. *W. G. Cochran.*

Silberstein, Ludwik. On two accessories of three-dimensional colorimetry. I. The probable error of colorimetric tensor components as derived from a number of color matchings. II. The determination of the principal colorimetric axes at any point of the color threefold. J. Opt. Soc. Amer. 36, 464–468 (1946).

The probability density of a color match falling at (x_1, x_2, x_3) is supposed to be given by $(*)$ $\exp\{-\sum g_{ik}x_ix_k\}$. The author shows in the first part how the most probable values for the unknowns g_{ik}, and the probable errors of these values, can be computed from n observations. The second part describes the familiar method of reduction of the quadratic form in $(*)$ to principal axes using the secular equation.

W. Feller (Ithaca, N. Y.).

Bachmann, W. K. Théorie des erreurs de l'observation des variables secondaires. Schweiz. Z. Vermessgswes. Kulturtech. 43, 21–27, 45–51 (1945). [MF 13999]

L'auteur appelle une variable primaire, respectivement secondaire, suivant que nous pouvons agir directement sur elle ou non. Il faut distinguer entre les observations et les mesures. Les dernières sont exemptes d'erreur, les premières sont toujours erronées. Seules les variables primaires peuvent être mesurées; les observations peuvent se rapporter aux deux sortes de variables. Dans chaque problème d'erreur "observation" et "mesure" s'enchaînent. La théorie des erreurs courante ne traite que le cas de l'observation des variables primaires. Pour les variables secondaires les opéra-

tions se compliquent très rapidement avec le nombre des variables. Il faut alors scinder le problème en plusieurs parties, contenant chacune un nombre inférieur de variables. L'auteur traite comme example numérique à deux variables le rétablissement d'un point de triangulation à l'aide de visées extérieures. Trois méthodes différentes donnent lieu à des ellipses d'erreur différentes. Il en résulte que la méthode d'observation doit être connue si l'on veut calculer les erreurs justement. *E. Bodewig* (La Haye).

Mathematical Biology

***Householder, Alston S., and Landahl, Herbert D. Mathematical Biophysics of the Central Nervous System.** Mathematical Biophysics Monograph Series, No. 1. Principia Press, Inc., Bloomington, Ind., 1944. ix+124 pp. $2.00.

The book is an exposition and an enlargement of work which, to a great extent, was done by the authors themselves towards the establishment of a mathematical neuropsychology. The basic structure which is the subject of the study is schematized in the nerve cells, the neurons, connecting the sense organ and the muscle. From a mathematical viewpoint the neuron is characterized by two functions of the time t, the excitation E and the inhibition I. No attempt is made to give a physicochemical interpretation of these two quantities. It is postulated that E and I follow a type of relationship reminiscent of chemical kinetics,

$$(1) \qquad dE/dt=a(f-E), \quad dI/dt=b(g-I),$$

where a, b are constant and f, g are functions of the intensity S of the stimulus. The functions $f(S)$ and $g(S)$ are assumed to be of a threshold type, so that $f(S)=g(S)=0$ for sufficiently small values of S. Besides this it is assumed that $f(\infty), g(\infty)$ are finite. On these assumptions the theory of a more realistic type of connection between the sense organ and the muscle can be built up. Such a connection consists of nets of neurons leading fundamentally to a system of equations of the above type. The general case of a spatially distributed neuron net in which each neuron has different response characteristics leads to an integral equation for the calculation of the global excitation and inhibition. An approximation of the functions f and g by two parallel straight lines joined by an oblique one reduces the study of neuron chains to finite difference equations. Consideration of time fluctuations of the thresholds introduces probabilistic concepts which are applied to an interpretation of sequences of stimuli and responses. The book ends with an outline of the approach of W. S. McCulloch and W. Pitts in which the response of the neuron is schematized in the all-or-none type of event. This makes an immediate application of the Boolean algebra possible. It is shown that this latter theory and the one based on equations (1) may be bridged together through a statistical approach in which the response of the neuron is made dependent on the occurrence, within a fixed time, of a certain number of statistically independent impulses.

I. Opatowski (Chicago, Ill.).

Rashevsky, N. Further contributions to a probabilistic interpretation of the mathematical biophysics of the central nervous system. Bull. Math. Biophys. **8**, 51–57 (1946). [MF 16901]

Opatowski, I. The probabilistic approach to the effects of radiations and variability of sensitivity. Bull. Math. Biophys. **8**, 101–119 (1946).

The notions of sensitive volume and control center are discussed assuming that the effects of radiation can be explained on the basis of random hits. The theory of Markov chains then becomes the main tool of the consideration. The mean life is calculated under simplifying assumptions, but assuming recovery. Methods for comparison of theory with observations are given.

W. Feller (Ithaca, N. Y.).

Koyenuma, Nobutsugu. Zur theorie der biologischen Strahlenwirkung. II. Phys. Z. **42**, 213–217 (1941).

[The first part appeared in Z. Physik **117**, 510–514 (1941); cf. also Z. Physik **120**, 185–211 (1943); these Rev. **4**, 201.] The author assumes that the chromosomes and genes have a crystalline structure and interprets the biological effects of radiation as an absorption process. Known physical laws then lead immediately to a relation between the temperature and the probability of scoring n harmful hits. *W. Feller* (Ithaca, N. Y.).

Rosenblatt, Alfred. On the theory of the X chromosomes in genetics. Revista Ci., Lima **48**, 3–17 (1946).

Defining a, b, c as "the dominant chromosome, the recessive chromosome and the deficient chromosome of the male," the author studies the structure of the nth descendant generation derived under random mating from an initial crossing bb, ac or ab, ac. *W. Feller* (Ithaca, N. Y.).

Malécot, G. Étude mathématique des populations "mendéliennes." I. Consanguinité pure. Ann. Univ. Lyon. Sect. A. (3) **4**, 45–60 (1941).

A systematic study of S. Wright's coefficient of inbreeding for the descendants of parents who are related to each other.

W. Feller (Ithaca, N. Y.).

Malécot, Gustave. La consanguinité dans une population limitée. C. R. Acad. Sci. Paris **222**, 841–843 (1946). [MF 16277]

Consider a finite vegetable population with possible self-fertilization, where the number of individuals is kept constant for all generations. A simple argument shows that the probability tends to one that any two homologous genes in the nth generation are derived from some common ancestor gene. It follows in particular that the population tends towards genetic homogeneity. Similar situations are shown to prevail in some other finite populations.

W. Feller (Ithaca, N. Y.).

Doss, Raouf. Sur la théorie mathématique de la croissance des organismes. Proc. Math. Phys. Soc. Egypt **2**, 29–43 (1946). [MF 16833]

From general principles the author derives a general equation of growth and some associated relations, from which classical equations follow on specializing the parameters. Growth resulting from the excess of assimilated over dissimilated material, and dissimilation being supposed proportional to the mass of the organ, the first task is the derivation of an expression for the rate of assimilation. This depends upon the rate x of supply of assimilable mate-

rial in such a way that, $\xi=\phi(x)$ being the rate of assimilation, if $x+y$ is supplied in unit time the rate becomes $\phi(x+y)=f[\phi(x), \phi(y)]$, where $f(\xi, \eta)$ is symmetric. For a prescribed $f(\xi, \eta)$, ϕ is shown to satisfy $\gamma(\xi, \eta)d\eta/dy=\partial f/\partial\xi$, with γ also symmetric.

The most general possible polynomial f has the form $f=\xi+\eta-k\xi\eta$ and in this case $\xi=k^{-1}(1-e^{-\sigma kx})$. For any organ of weight p_i, the associated σ_i, the "coefficient of assimilation," depends on p_i and on $p=\sum p_i$, the total weight of all the organs. In this manner one derives the growth equation

$$dp_i/dt=-\epsilon_i' p_i+\epsilon_i p_i\{1-e^{-(\sigma_i x)/(\sigma_i p)}\}.$$

For small x this implies a classical empirical relation, $p_i=b_i p^{a_i}e^{-h_i t}$.

Next, having regard for the interaction of the digestive glands p_i and the action of the blood p_s in supplying material, the author develops an expression for x as a function of the p_i and p_s, on the assumption of an adequate external source of nutriment. This permits the derivation, finally, of an expression for the rate of growth dp/dt which the author writes expanded in fractional powers of p. As applied to man, the agreement with empirical data is satisfactory.

A. S. Householder (Oak Ridge, Tenn.).

Diaz, J. Gallego. A new mathematical theory of the division of cells. Gaz. Mat., Lisboa **7**, no. 29, 12–13 (1946). (Spanish)

This is an effort to describe the course of cellular division on the assumption that every particle of the cell is attracted to each centrosome by a force inversely proportional to its distance therefrom. The mechanism separating the centrosomes is not discussed. The author derives the Cassini oval as the figure of equilibrium with lines of force forming hyperbolas through the centrosomes. *A. S. Householder.*

Szymański, Piotr. Essai sur la théorie mathématique des sensations de couleur. II. Mathematica, Timişoara **22**, 13–40 (1946).

Part I appeared in vol. **21**, 69–96 (1943); these Rev. **5**, 52.

TOPOLOGY

Tietze, Heinrich. Bemerkungen über verknotete und verkettete Linien. II. Vorgeschriebene singuläre Primzahlen für eine Simony-Figur und für ihr Spiegelbild. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1943**, 265–268 (1944).

[For part I see the same S.-B. **1942**, 53–62 (1942); these Rev. **5**, 151.] The author now derives conditions for the existence of torus knots $[2, n]$ with prescribed singular prime numbers of the knot and prescribed singular prime numbers of the reflected knot. (A singular prime is one with Minkowski unit -1.) *H. Samelson* (Ann Arbor, Mich.).

Ratib, Ismail. Sur le problème des quatres couleurs. Proc. Math. Phys. Soc. Egypt **2**, 49–59 (1944). [MF 16834]

In attempting to prove that every regular map, irreducible with respect to four colors, must contain at least one pentagon adjacent to two other pentagons, the author finds that this must occur unless the map contains one of certain specified "stubborn" configurations. In this paper he proves that three of these "stubborn" configurations cannot occur. He uses the method of chains and has to consider several hundred special possibilities. *P. Franklin.*

Kagno, I. N. Linear graphs of degree ≤ 6 and their groups. Amer. J. Math. **68**, 505–520 (1946).

A graph is a set of n points called vertices, certain pairs of which are joined by segments called edges. Only combinatorial properties are considered, so a graph is completely determined by its scheme of edges (in any order); for example, the triangle abc is given by $[ab, ac, bc]$. The group of a graph is defined to consist of all permutations of the vertices that preserve the scheme of edges; for example, the group of the triangle is the symmetric group "(abc)all." No graph has a cyclic group (with $n>2$); for, the cyclic permutation $(ab\cdots yz)$ cannot occur without the "reflection" $(az)(by)\cdots$. Similarly, no graph has an alternating group. The author restricts consideration to connected graphs in which every vertex belongs to at least three edges; thus the only admissible graph with $n=4$ is the tetrahedron, whose group is $(abcd)$all. He finds five admissible graphs with $n=5$, eighteen with $n=6$, and obtains their groups, all of which have order greater than 1. He also describes a graph with $n=7$ whose group is merely the identity. Finally, he considers the complete list of permutation groups of degree less than 7 and shows which of them belong to graphs (of the same or greater degree). *H. S. M. Coxeter.*

Kreweras, Germain. Peut-on former un réseau donné avec des parties finies d'un ensemble dénombrable? C. R. Acad. Sci. Paris **222**, 1025–1027 (1946). [MF 16379]

The set F of all finite subsets of a denumerable set D is regarded as the set of vertices of a graph G_F in which two vertices $A, B \varepsilon F$ are joined (by an edge) if and only if the sets A, B are disjoint. It is proved that a finite or denumerable graph R is isomorphic to a subgraph of F if and only if, for each vertex of R, the vertices of R can be partitioned as $R=R_a+R_1+\cdots+R_n$, where R_a consists of all vertices joined to a and where no two vertices of any one R_i are joined $(i=1, \cdots, n)$. *S. MacLane* (Cambridge, Mass.).

Paintandre, Roger. Sur une décomposition de la frontière d'un ensemble. C. R. Acad. Sci. Paris **223**, 121–123 (1946).

Let I, E, and S be the interiors of a set A, of its complement A', and of its frontier F, respectively. It is shown that $F-S$ is covered by its generally nonvoid intersections with $I^-E^-S^-$, $I^-E^-S^{-\prime}$, $I^{-\prime}E^{-\prime}S^-$, $I^{-\prime}E^-S^{-\prime}$, $I^-E^{-\prime}S^-$, and $I^-E^{-\prime}S^{-\prime}$, constituting, with S, the author's decomposition of F; with I, E, and S, a decomposition of the space. Regular and other sets A are characterized by the vacuity of certain components. Each of the sets $A^-, A^{-\prime}, A^{-\prime-}, \cdots, A^{\prime-}, A^{\prime-\prime}, \cdots$ can be expressed uniquely as a union of suitable members of this decomposition. [p. 122, l. 15: read "$=$" for "$\neq$"; p. 123, l. 6: read "eis" for "fis"; l. 8: add "+Aeis".] *R. Arens* (Princeton, N. J.).

Bing, R. H. The Kline sphere characterization problem. Bull. Amer. Math. Soc. **52**, 644–653 (1946).

The author gives the first proof of a proposition conjectured by J. R. Kline: in order that a nondegenerate Peano space be a 2-sphere, it is necessary and sufficient that it be cut by every 1-sphere but by no 0-sphere. The proof is made on the basis of earlier results of Moore and Zippin. [A detailed historical and bibliographical discussion of the problem is given by E. R. van Kampen, Duke Math. J. **1**, 74–93 (1935).] *A. D. Wallace* (Philadelphia, Pa.).

Bing, R. H. Concerning simple plane webs. Trans. Amer. Math. Soc. **60**, 133–148 (1946).

Let W be a compact plane continuum. Then W is a simple web provided there exist two upper semicontinuous collections of continua G, H covering W such that (1) G, H are dendrons, (2) if $g \in G$, $h \in H$ then dim $(g \cap h) = 0$. The principal theorem of the paper states that W is a simple web if and only if the complement of every countable set is connected and locally connected. Making use of this result, the author [Bull. Amer. Math. Soc. **51**, 674–679 (1945); these Rev. **7**, 136] showed earlier that condition (2) implies condition (1). In addition, it is proved, among other things, that, if W is a simple web, then there are a dendron D in W and monotone retractions R_1, R_2 such that for $a, b \in D$ we have dim $(R_1^{-1}(a) \cap R_2^{-1}(b)) = 0$. *A. D. Wallace.*

Bing, R. H. Generalizations of two theorems of Janiszewski. II. Bull. Amer. Math. Soc. **52**, 478–480 (1946). [MF 16805]

This paper gives minor extensions and corrections to part I [same Bull. **51**, 954–960 (1945); these Rev. **7**, 216]. *J. H. Roberts* (Durham, N. C.).

Bing, R. H. Sets cutting the plane. Ann. of Math. (2) **47**, 476–479 (1946).

This paper contains six theorems concerning cutting and disrupting in the plane, related to work of Kuratowski [Monatsh. Math. Phys. **36**, 77–80 (1929)] and Eilenberg [Fund. Math. **24**, 160–176 (1935); **26**, 61–112 (1936)]. A point set R disrupts A from B in W if there exists an arc AB in W but each such arc hits R. Theorem: if the connected set K does not disrupt the point A from the point B, then there exists an arc AB such that, if a point belongs to $AB \cdot \bar{K}$, it belongs to every continuum in the complement of K that contains $A+B$. *J. H. Roberts* (Durham, N. C.).

Kapuano, I. Sur les corps de nombres à une dimension distincts du corps réel. Rev. Fac. Sci. Univ. Istanbul (A) **11**, 30–39 (1946). (French. Turkish summary) [MF 16713]

The existence of a one-dimensional, algebraically closed subfield of the complex number system, containing no real transcendental numbers, is established. The method involves transfinitely successive adjunction of elements of certain frontier sets of the plane and resembles that used independently by Dieudonné [C. R. Acad. Sci. Paris **221**, 396–398 (1945); these Rev. **7**, 215] for constructing a proper dense connected subfield of the complex number system. *R. Arens* (Princeton, N. J.).

Knaster, Bronisław. Sur un problème de P. Alexandroff. Fund. Math. **33**, 308–313 (1945). [MF 16896]

The equivalence of the following three properties of a metric separable space X is proved: (1) X is homeomorphic to a connected metric separable space with one point removed, (2) X is homeomorphic to a proper open subset of a connected metric separable space, (3) there exists in X a descending sequence $\{F_i\}$ of closed sets with a vacuous intersection such that each set F_i intersects every open and closed subset of X. *S. Eilenberg* (Bloomington, Ind.).

Barbashin, E. On σ-coverings of spaces. Rec. Math. [Mat. Sbornik] N.S. **18(60)**, 423–428 (1946). (Russian. English summary)

Soit R un espace métrisable et séparable. Soit S un système d'ensembles ouverts couvrant R et tel que chaque sous-ensemble d'un ensemble de S appartient à S, et la somme de deux ensembles de S, dont l'intersection est vide, appartient à S. Soit $M \subset R$ et soit k le plus petit nombre cardinal tel qu'il y a un recouvrement de M par k éléments de S. Alors k est nommé la catégorie (S) de M, $K_S(M)$. Exemple: la catégorie de Lusternik-Schnirelmann. Autre exemple: on prend pour S le système de tous les ensembles ouverts dont les components sont de diamètre inférieur à ϵ. Soit T un recouvrement d'un ensemble A. Soit m le plus grand nombre tel qu'il y a un point appartenant à k ensembles T. Alors m est appelé l'ordre de T.

(1) Si $K_S(M)$ est finie, alors dim $M \geqq K_S(M) - 1$. (2) Si $K_S(M)$ est finie, alors il y a un élément de S partout dense en R. (3) Si $K_S(M)$ et dim M sont finies, alors il y a un G ouvert en M tel que $K_S(G) = 1$ et dim $(M-G) <$ dim M. (4) Si $K_S(A)$ est finie, alors l'ordre minimal des recouvrements de A par des ensembles de S est égal à $K_S(A)$. (5) Si A et B sont fermés dans R, $K_S(A) \leqq n$, $K_S(B) \leqq n$ et dim $AB \leqq n-2$, alors $K_S(A+B) \leqq n$. (6) La propriété de compacta d'être d'une catégorie (S) fixe est inductive. (7) Chaque compactum contient un continuum de la même catégorie (S). (8) Chaque compactum R de dimension finie satisfaisant à $K_S(R) =$ dim $R+1$ contient une multiplicité de Cantor de la même catégorie (S) et de la même dimension. *H. Freudenthal* (Amsterdam).

Mazurkiewicz, Stefan. Recherches sur la théorie des bouts premiers. Fund. Math. **33**, 177–228 (1945).

L'auteur expose la théorie des bouts premiers de B. Kaufmann [Math. Ann. **103**, 70–144 (1930); **106**, 308–342 (1932)] et une théorie nouvelle se rattachant à celle-là. Les bouts premiers introduits par l'auteur admettent une définition plus élégante que ceux de Kaufmann et conduisent à une fermeture qui jouit des propriétés plus satisfaisantes.

L'auteur considère une variété non-compacte R avec une métrique naturelle, c'est-à-dire où la distance de deux points a, b est égale à la limite inférieure des continus joignant a et b. Une suite est nommée asymptotique, si elle ne possède pas de points d'accumulation dans R. Parmi ces suites on distingue les suites α satisfaisantes au critère de convergence de Cauchy et les suites β qui ne contiennent aucune suite partielle α. Deux suites a_n et b_n de ces types sont dites conjuguées, s'il y a des arcs simples $a_n b_n$ tels que toutes les suites $c_n \in a_n b_n$ sont du même type α ou β. Une suite est nommée normée, si elle est conjuguée à chaque suite partielle.

La définition fondamentale est la suivante. L'ensemble W est dit une frontière topologique de R, si (1) R est un sous-espace de $R+W$, (2) $W \subset \bar{R}$, (3) $R+W$ est compact et métrisable. Théoreme fondamental. Il y a une frontière topologique W_0 satisfaisante aux conditions: (4) si la suite $a_n \in R$ converge vers le point a de W_0, chaque suite conjuguée à a_n converge également vers a; (5) W_0 est plus faible que chaque autre frontière satisfaisante à (1)–(4) et non-identique avec W_0; c'est-à-dire l'ensemble des suites de R convergentes dans $R+W_0$ est minimal. Les éléments de W_0 sont les bouts premiers d'après la définition nouvelle.

Pour construire ses bouts premiers, Kaufmann avait considéré les classes de suites normées conjuguées, puis il les avait élargies par un procédé de fermeture; c'est ce procédé qui est remplacé par un procédé plus satisfaisant dans le mémoire actuel. L'espace $R+W_0$ est un espace péanien et W_0 est en chaque point régulièrement accessible dans $R+W_0$. Un exemple fair voir la différence entre la théorie de Kaufmann et celle de l'auteur. *H. Freudenthal.*

de Possel, René. Sur un espace sans base dénombrable qui est complètement ordonné et dont tout intervalle fermé est homéomorphe à un intervalle numérique. C. R. Acad. Sci. Paris **222**, 1202–1203 (1946). [MF 16729]

This space E is the Cartesian product of the class of finite and countable ordinals times the interval $[0, 1)$, with dictionary order. The properties of E derived here are fairly well known. In particular, that mentioned in the title was proved by C. W. Vickery [Tôhoku Math. J. **40**, 1–26 (1935), p. 18, theorem 2]. *R. Arens* (Princeton, N. J.).

Sierpiński, Wacław. Sur les espaces (V) de M. Fréchet denses en soi. Fund. Math. **33**, 174–176 (1945). [MF 16880]

It is proved that any set X which is an "espace (V)" in the sense of Fréchet and is dense-on-itself can be made into another "espace (V)" X^* in which the open sets are the dense-on-themselves sets of X and in which the dense-on-themselves sets are the open sets of X. *R. Arens.*

Ferrari, Esther. On general topological spaces. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 183–189 (1946). (Spanish) [MF 16921]

This paper states the results of the first chapter of the author's doctoral dissertation. It is concerned with the minimum set of postulates needed to prove various results about abstract spaces (L), (V), (D_0), etc. (Fréchet). Use is made of postulates of Riesz, Kuratowski, and Hausdorff. Perhaps the most interesting result is the following analogue of Urysohn's theorem concerning the extension of the domain of definition of a continuous function. In a (D_0) space (such spaces are not necessarily normal), if C is a complete subset of D_0 (that is, considered as a space, C is complete) and $f(x)$ is a real positive function defined and continuous over C, then there exists an upper semicontinuous function $F(x)$ defined over D_0 such that $F(x)=f(x)$ for $x \varepsilon C$. *J. H. Roberts* (Durham, N. C.).

Doss, Raouf. Sur la condition de régularité pour l'écart abstrait. C. R. Acad. Sci. Paris **223**, 14–16 (1946).

The author discusses abstract spaces E on which a regular écart is defined with values in an ordered set S, where S has a first element 0 but no second element. The écart $(a, b) \varepsilon S$ is to satisfy the usual requirements (1) $(a, b)=0$ if and only if $a=b$; (2) $(a, b)=(b, a)$; (3) for every $\xi \varepsilon S$ such that $\xi \vdash 0$, there exists an element $\varphi(\xi) \vdash 0$ such that $(a, b) \dashv \varphi(\xi)$ and $(b, c) \dashv \varphi(\xi)$ imply $(a, c) \dashv \xi$. The author states that, if E is not metrizable, the function φ in (3) may be taken to be the identity, $\varphi(\xi)=\xi$. A proof is sketched. He deduces from this result that any nonmetrizable abstract space with a regular écart is a completely regular topological space with a complete family of open and closed neighborhoods for each point. From this result, one may infer the following consequence: any completely regular topological space having a completely ordered uniform structure is either metrizable or totally disconnected. *E. Hewitt* (Bryn Mawr, Pa.).

Fréchet, Maurice. De l'écart numérique à l'écart abstrait. Portugaliae Math. **5**, 121–131 (1946).

Let S be a linearly ordered set with a nonisolated first element 0 and on which there is defined a function φ, $\varphi(\xi) \varepsilon S$ for $\xi \varepsilon S$, such that $\varphi(\xi) \neq 0$ if $\xi \neq 0$ and $\varphi(\xi) \to 0$ as $\xi \to 0$. Let X be any set and suppose there is a function ρ, $\rho(x, y)=\rho(y, x) \varepsilon S$ if $x, y \varepsilon X$; $\rho(x, y)=0$ precisely when $x=y$; and such that $\rho(x, y)$, $\rho(y, z) \leqq \xi$ implies $\rho(x, z) \leqq \varphi(\xi)$. Then (S, φ, ρ) is a symmetric regular abstract écart (s.r.é.) for X and X is an s.r.é.-space. In the terminology of A. Weil, with whose theory the author desires the preceding to be contrasted, one has here a uniform structure for X with a linearly ordered base. The author desires to show, by examples, that known proofs for metric spaces can be extended more easily to s.r.é.'s than to the uniform structures of Weil. In a supplementary note, the author asks whether there exists any uniform structure which cannot be replaced by an s.r.é. giving the same topology. [The answer is: yes. An extension of a familiar proof [Alexandroff-Hopf, Topologie I, Springer, Berlin, 1935, p. 68] shows that only a completely normal space can be made into an s.r.é.; but, of course, any completely regular space has a uniform structure in the sense of Weil.] *R. Arens* (Princeton, N. J.).

Gorciu, V. G. The Moore-Smith convergence in topology and the theory of filters. Bull. Math. Soc. Roumaine Sci. **46**, 141–143 (1944). [MF 16517]

The author shows that the convergence of directed sets in topological spaces can be described in terms of the concept of "filter" introduced by H. Cartan. [Cf. J. W. Tukey, Convergence and Uniformity in Topology, Ann. of Math. Studies, no. 2, Princeton University Press, 1940, chap. III; these Rev. **2**, 67.] *G. Birkhoff* (Cambridge, Mass.).

Monna, A. F. Sur l'approximation de fonctions abstraites. Nederl. Akad. Wetensch., Proc. **49**, 404–408 = Indagationes Math. **8**, 259–263 (1946). [MF 16822]

Let E be a metric space; let $\Delta(a, b)$ be the lower bound of those numbers r such that a and b can be connected by an r-chain (presumably a finite sequence of points each within r of its predecessor). Theorem 4. If E is separable and b a point of E, then the set of numbers $\Delta(a, b)$ for a in E is countable. The proof uses theorem 3: if f_n converges pointwise to f and if for each n the set of numbers $\Delta(f_n(x), b)$ is countable, then the set of numbers $\Delta(f(x), b)$ is countable. The proof is given for any non-Archimedean metric. Theorems 1 and 2 are, essentially, the obvious conditions that a function f with values in E is uniformly (pointwise) approximable by functions taking only a finite number of values if and only if the set of all points $f(x)$ is totally bounded (separable). There are some obvious errors, particularly the consequence of theorem 1 and the remark after theorem 2; none of these affects the validity of the four main theorems. *M. M. Day* (Urbana, Ill.).

Borsuk, Karol. On the decomposition of manifolds into products of curves and surfaces. Fund. Math. **33**, 273–298 (1945).

A decomposition of a space E is a collection $E_1, E_2, \cdots, E_m$ of spaces whose Cartesian product $E_1 \times E_2 \times \cdots \times E_m$ is homeomorphic to E; the sets E_i are called divisors of E. Every set E has itself and the space which consists of one point as trivial divisors; if these are the only divisors, then E is called prime. Any nondegenerate finite dimensional compact space with a finite number of components can be decomposed into nondegenerate prime sets. In general such a decomposition is not unique: for example, there exist bounded 3-dimensional manifolds which can be decomposed in several ways into the product of prime bounded manifolds. [One such was given by J. H. C. Whitehead, Ann. of Math. (2) **41**, 825–832 (1940); these Rev. **2**, 73.] The author has previously proved that no polyhedron of any dimension can have more than one decomposition into a product of (nondegenerate) prime sets of dimension not

exceeding 1 [Fund. Math. **31**, 137–148 (1938)]. In view of the example quoted, this result is, for polyhedra, the best possible.

In the present contribution the author shows that no closed n-dimensional manifold (that is, a space locally homeomorphic to Euclidean n-dimensional space) can have more than one decomposition into a product of (nondegenerate) prime sets of dimension not exceeding 2. Whether this is the best possible result for closed manifolds is, of course, an open question. In the course of deriving the main theorem the decompositions of an open subset of Euclidean n-dimensional space are investigated; in particular, it is shown that the only 1- and 2-dimensional divisors of Euclidean n-dimensional space are the 1- and 2-dimensional Euclidean spaces, and that the 1- and 2-dimensional divisors of a closed n-dimensional manifold are closed 1- and 2-dimensional manifolds. *R. H. Fox* (Princeton, N. J.).

Ephrämowitsch, V. A. On non-decomposibility into a topological product. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 470–471 (1945).

If the Poincaré polynomial of an n-dimensional separable metric space is irreducible and if the n-dimensional Betti number is positive, then the space cannot be a topological product. The coefficient group can be the integers or the integers mod p, where p is prime. An immediate consequence is that no 2-dimensional manifold, with the possible exceptions of the torus and the Klein bottle, is a topological product. *E. Begle* (New Haven, Conn.).

Braconnier, Jean, et Colmez, Jean. Sur les groupes d'homéomorphismes d'un espace complètement régulier. C. R. Acad. Sci. Paris **223**, 230–232 (1946).

The authors formulate conditions sufficient to insure that a group G of homeomorphisms of a space X may be made into a topological group such that $g(x)$ depends continuously on g and x jointly (that is, admissible topology for G). These are satisfied when X is locally compact. *R. Arens.*

Young, Gail S., Jr. The introduction of local connectivity by change of topology. Amer. J. Math. **68**, 479–494 (1946).

The author considers a way of changing the topology of a given topological space S. If $\{G_\lambda\}$ is a family of subsets of S, then a point $p \in S$ is a G-limit point of $M \subset S$ if and only if every open set containing p contains a set G_λ such that $G_\lambda \cap M \neq 0 \neq G_\lambda \cap M'$. It is obvious that the topology defined in this way is an expansion of the original. The author is particularly concerned with the cases in which $\{G_\lambda\}$ is the family of all connected subsets of S, all locally connected subsets of S, all arcs in S or all rectifiable arcs in S. Retention of (1) metrizability, (2) the property of being a Moore space, (3) metric completeness, and the introduction of local connectedness, under the passage to a G-topology, are discussed. A "generalized dendrite" is defined and discussed and applications of the theory of G-topologies are given. *E. Hewitt* (Bryn Mawr, Pa.).

Leray, Jean. L'anneau d'homologie d'une représentation. C. R. Acad. Sci. Paris **222**, 1366–1368 (1946).

Leray, Jean. Structure de l'anneau d'homologie d'une représentation. C. R. Acad. Sci. Paris **222**, 1419–1422 (1946).

A "bundle" of groups in a topological space X is a function which with every closed subset F of X associates a group B_F and with every inclusion $F' \subset F$ a homomorphism $B_F \to B_{F'}$ subject to the usual transitivity condition. Moreover, B_F should be the trivial group if F is vacuous. A bundle of groups can be used as a coefficient system for homology and cohomology in the space X. Let $f: X \to Y$ be a continuous map and let p, q be integers. For each closed set $F \subset Y$, B_F is defined as the pth cohomology group of $f^{-1}(F)$ with coefficients in a ring A. This gives a bundle of groups in Y with respect to which the qth cohomology group is constructed. The resulting group is called the (p, q)-module of f over A.

The second paper enters in more detail into the structure of this new group and states without proofs a number of applications. *S. Eilenberg* (Bloomington, Ind.).

Chogoshvili, G. On the duality law in normal spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 233–235 (1945). [MF 16660]

A duality theorem, of the Alexander-Kolmogoroff type, is described for the cohomology groups of certain pairs of complementary subsets of a normal space. When the space is locally bicompact and one of the sets is closed, the theorem reduces to the well-known Kolmogoroff duality theorem [cf. Alexandroff, Trans. Amer. Math. Soc. **49**, 41–105 (1941), in particular, p. 86; these Rev. **2**, 323]. *E. Begle* (New Haven, Conn.).

Chogoshvili, G. The duality law for retracts. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 91–94 (1946).

The author has previously [C. R. Acad. Sci. Paris **221**, 15–17 (1945); these Rev. **7**, 216] extended the Pontrjagin duality theorem from the case of a closed subset A of the n-sphere S^n to the case where $S^n - A$ is any infinite polyhedron, not necessarily open. Here a further extension is made to the case where $S^n - A$ is any set which is a retract of some open set in S^n containing it. *E. Begle.*

Ostrowski, Alexandre. Sur l'inverse d'une transformation continue et biunivoque. C. R. Acad. Sci. Paris **223**, 229–230 (1946).

A proof, based on the Jordan separation theorem, of the theorem that, if the inverse of a continuous mapping into R_n of an open subset of R_n is single-valued, it is continuous. *P. A. Smith* (New York, N. Y.).

Lacombe, Daniel. Sur les champs vectoriels dont la normalité à quelque droite, réalisée en un point, s'étend de ce fait à toute la droite. Revue Sci. **83**, 167–169 (1945). [MF 16984]

The author studies vector fields in domains of 3-space with the following property: if the field is normal to an arbitrary line D at one point of D, then it is normal to D at every point of D. Such a field is proved to be continuous. A plane is called an indifferent polar plane if the field is normal to the plane at every point of the plane. If there exists at least one indifferent polar plane, then the field is shown to consist of the normals to a one-parameter family of planes; if there exists no such plane the field is shown to be a field of moments, as defined in mechanics or line geometry. *H. Samelson* (Ann Arbor, Mich.).

Frankl, Felix. To the topology of the three-dimensional space. Rec. Math. [Mat. Sbornik] N.S. **18**(60), 299–304 (1946). (Russian. English summary)

The author constructs a two-complex $K = A + B + C$, with vanishing homology and fundamental groups, as follows. Let a, b, c denote simple oriented one-circuits (regarded also

as continuous paths) originating from a common point. Then A, B, C denote closed two-cells with singular boundaries, these being, respectively, $aba^{-1}cac^{-1}b^{-1}$, $bcac^{-1}b^{-1}aba^{-1}$ and $cabc^{-1}cac^{-1}b^{-1}$ (the superscript relates to the inverse path). This is an example of a type of "canonical skeleton" to which a three-manifold with sphere boundary may be continuously deformed. It serves the author as a counter-example to a seemingly plausible conjecture.

L. Zippin (Flushing, N. Y.).

Kuratowski, Casimir. Théorèmes sur l'homotopie des fonctions continues de variable complexe et leurs rapports à la théorie des fonctions analytiques. Fund. Math. **33**, 316–367 (1945). [MF 16718]

The purpose of the paper is to exhibit a strong connection between the topology of subsets of the 2-sphere S_2 (understood as the complex plane with a point at infinity) and the theory of analytic functions. The method is the one used by the reviewer in his thesis [Fund. Math. **26**, 61–112 (1936)], with the modification that instead of using mappings into the circumference $S_1=\{|z|=1\}$ the author maps his spaces into the punctured sphere $P=\{z\neq 0,\ z\neq\infty\}$. Since P is the Cartesian product of S_1 and an open interval, this change has little topological significance; however, it is very helpful in bringing the statements into closer analogy with analytic functions. The crucial fact is that a map $f:X\to P$ (or $X\to S_1$) has a continuous logarithm if and only if f is homotopic to a constant. Dividing the multiplicative groups of all maps f by those which have a continuous logarithm, the group $B(X)$ is obtained; this group is closely related with the one-dimensional homology properties of X.

The first two chapters constitute a rewrite of a portion of the reviewer's thesis [loc. cit.], with S_1 replaced by P and with many proofs improved. In the third chapter the following analogue of Runge's theorem is proved: every map $f:G\to P$, where G is an open subset of S_2, is of the form $f(x)=\lim r_n(x)e^{U_n(x)}$ with r_n rational and with $U_n:S_2\to S_2$, $U_n(x)\neq\infty$. The convergence is uniform on every compact subset of G. A similar analogue of Weierstrass's theorem is established. The fourth chapter deals with the multiplicity $\mu_F f$ of a function f (as above) with respect to a set F which is open and closed in S^2-G; $\mu_F r_n$ is defined to be the algebraic number of zeros and poles of r_n in F, and $\mu_F f=\lim \mu_F r_n$. With the aid of this concept the author proves a duality theorem between $B(G)$ and a suitably defined analogue of the 0th homology group of S^2-G. The last chapter compares the concept of multiplicity with that of the index of a point with respect to a curve. *S. Eilenberg.*

Whitehead, George W. On products in homotopy groups. Ann. of Math. (2) **47**, 460–475 (1946).

If X is a space and $F^p(X)$ is the function space of maps of a p-cell in X carrying the boundary into a fixed point, Hurewicz proved that the nth homotopy group $\pi_n(X)$ of X is isomorphic to $\pi_{n-p}(F^p(X))$ for $p=0, 1, \cdots, n-1$. It is to be expected that the standard operations on the elements of the homotopy groups of X should translate, under these isomorphisms, into operations on the homotopy groups of these function spaces. The author deduces explicit formulas for the operations which correspond to the J. H. C. Whitehead product [Ann. of Math. (2) **42**, 409–428 (1941); these Rev. **2**, 323] and the Freudenthal Einhängung [Compositio Math. **5**, 299–314 (1937)]. These formulas have the advantage that they are expressed entirely in terms of the natural operations of boundary and induced homomorphism. The known properties of the latter operations translate back into properties of products and of Einhängung. An interesting new result obtained in this way is that the Einhängung of a product is always zero. Several generalizations of the Whitehead products are defined and studied. *N. E. Steenrod.*

Whitehead, George W. A generalization of the Hopf invariant. Proc. Nat. Acad. Sci. U. S. A. **32**, 188–190 (1946). [MF 16861]

If S^p and S^q are spheres of dimensions p and q with a single point in common, the author shows that the nth homotopy group $\pi_n(S^p\cup S^q)$ decomposes naturally into a direct sum $\pi_n(S^p)+\pi_n(S^q)+\pi_n(S^{p+q-1})$ when

$$n<p+q+\min(p,q)-3.$$

In the case $p=q=r$, this leads to a homomorphism H of $\pi_n(S^r)$ into $\pi_n(S^{2r-1})$ for each $n<3r-3$. It is a generalization of the Hopf invariant [Fund. Math. **25**, 427–440 (1935)] in the following sense: if $n=2r-1$ and $\alpha\in\pi_{2r-1}(S^r)$ has Hopf invariant h, then $H(\alpha)$ is h times a generator of the infinite cyclic group $\pi_{2r-1}(S^{2r-1})$. Since $H(\beta)\neq 0$ implies $\beta\neq 0$, a tool is thus provided for showing that certain maps of S^n on S^r are essential. The author obtains in this way the existence of essential maps for pairs $(n, r)=(12, 6)$, $(14, 7)$, $(8k, 4k)$ and $(16k+2, 8k)$. *N. E. Steenrod* (Ann Arbor, Mich.).

Manning, Rhoda. Open and closed transformations. Duke Math. J. **13**, 179–184 (1946).

A single-valued transformation is said to be closed if it preserves closure and to be exterior if it is closed and continuous. The author characterizes an exterior transformation as one which carries the closure of each subset of a space into the closure of the transform of that subset. A closed transformation is characterized as one that it is closed on every inverse set. For $f(A)=B$ closed, sufficient conditions are given (1) that A be compact, (2) that f be continuous. A principal result is as follows. In order that a closed transformation $f(A)=B$ be open it is necessary and sufficient that, for each sequence (y_n) in B converging to a point y in B, $\lim f^{-1}(y_n)=(f^{-1}(y))$. [See Eilenberg, Fund. Math. **24**, 160–176 (1935), in particular, p. 174.] After investigating conditions for homeomorphism of the original and the image set, the author concludes with a treatment of the effect of open weakly nonalternating transformations on arcs and simple closed curves. *W. W. S. Claytor.*

Floyd, E. E. On the extension of homeomorphisms on the interior of a two cell. Bull. Amer. Math. Soc. **52**, 654–658 (1946).

This paper considers homeomorphisms $f(I)=R$, where I is the interior of a two cell with boundary C and R is a plane bounded region with boundary $F(R)$. It is first shown that if f is uniformly continuous then the usual continuous extension $g(\bar{I})=\bar{R}$ has the property that $g(C)=R$ is a nonalternating transformation. The set A is a cut into R if A is contained in R and there exists a point x in $F(R)$ such that $A+x$ is an arc. The point x is called the end of the cut A. The transformation f possesses property P if both the following conditions hold: (1) A is a cut into R implies $f^{-1}(A)$ is a cut into I; (2) A_1 and A_2 being cuts into R with distinct ends implies that $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are cuts into I with distinct ends. It is shown that, for regions R with locally connected boundaries $F(R)$, the transformation f will have property P if and only if there exists a continuous extension $g(\bar{I})=\bar{R}$, $g=f$ on I, such that $g(C)=F(R)$ is light and nonalternating.

It has previously been shown [see Carathéodory, Conformal Representation, Cambridge University Press, 1932, pp. 82–85] that if I is the interior of the unit circle while f is any one-to-one conformal map then f possesses property P. In the light of the theorem stated above this has several corollaries, including the Osgood-Carathéodory theorem on conformal mapping.

Among the lemmas proved by the author the following appears to have independent interest. A necessary and sufficient condition that the boundary $F(R)$ of a bounded simply connected plane region R is locally connected is that $R+x$ is strongly arcwise connected at x for every point x in $F(R)$. *D. W. Hall* (College Park, Md.).

Eilenberg, Samuel, and Montgomery, Deane. Fixed point theorems for multi-valued transformations. Amer. J. Math. 68, 214–222 (1946). [MF 16417]

The following theorem is proved. Let M be an acyclic absolute neighborhood retract and $T: M \to M$ a continuous multivalued function such that, for each $x \in M$, $T(x)$ is acyclic. Then for some x, $x \in T(x)$. (The multivalued function T is continuous if, whenever $x_n \to x$, $y_n \to y$, $y_n \in T(x_n)$, then $y \in T(x)$.) The theorem contains as special cases the fixed point theorems of Kakutani for convex sets and of Wallace for trees. It also implies a generalization of the Lefschetz fixed point formula. The proof is first made for a finite complex and is then carried over to the general case by means of an imbedding theorem of Borsuk. *E. Begle.*

Birkhoff, George D., and Lifshitz, Jaime. Some transformations in dynamics without periodic elements. Publ. Inst. Mat. Univ. Nac. Litoral 6, 3–14 (1946). (Spanish) [MF 16911]

Let A be an annulus with bounding circles C_1 and C_2. Let R and U be homeomorphisms of A onto A such that $R(C_i)=C_i$, $U(C_i)=C_i$, $i=1, 2$, both R and U invert the order of points on C_1 and C_2 and $R^2=U^2=I$, where I is the identity. In the present paper a study is made of the transformation $T=RU$ under the assumption that there are no points of A which are periodic under T. It is first shown that each of the transformations R and U is topologically equivalent to a reflection and that each leaves invariant the points of two simple nonintersecting arcs which connect C_1 and C_2. With the additional hypothesis that T is area preserving, it is shown that the nonexistence of periodic points is equivalent to the existence of an irrational number σ which is the average angular rotation of every point of A under the iterates of T. It follows that there is a one-to-one correspondence between certain continua filling out A and the points of a circle Γ such that the transformation of Γ onto itself corresponding to T is a rotation through the angle σ. There are derived some added properties concerning invariant curves. *G. A. Hedlund.*

NUMERICAL AND GRAPHICAL METHODS

★A Manual of Operation for the Automatic Sequence Controlled Calculator, by the Staff of the Computation Laboratory. The Annals of the Computation Laboratory of Harvard University, vol. 1. Harvard University Press, Cambridge, Mass., 1946. xiii+561 pp. (17 plates) $10.00.

A hundred years ago Charles Babbage was working on an "analytical engine," which was to contain a "store" for holding numbers and a "mill" in which algebraical operations could be performed on these numbers according to instructions conveyed by punched cards. This is precisely what the new IBM automatic sequence controlled calculator does, except that the instructions are given by continuous punched tape rather than by separate punched cards. Its constructors have had advantages not enjoyed by Babbage, namely, standardised parts, modern tooling, electrical circuits and a team of engineers who have already had experience and success in the world of calculating machines. The principle is that of Babbage; the execution is that of IBM engineers and of H. H. Aiken.

Much of the detail in the manual is, naturally, of interest only to those using the machine or contemplating calculating machine design but the general description is of more universal interest. The "store," in the Babbage sense, consists mainly of 72 24-digit counters, of which one digit is reserved as a sign indicator. Besides these there are 60 switch-set counters of similar capacity for holding constants or other data that may be "called" from time to time. Data (for example, tabular functions) may also be entered into the machine from punched tapes or from Hollerith cards. The principal components of the "mill" are for multiplying and dividing, and for evaluating a logarithm or antilogarithm (or exponential) or sine (or cosine). Twenty-one-figure logarithms are found by dividing by four factors, one in each of the ranges 2, ···, 9, 1.1, ···, 1.9, 1.01, ···, 1.09 and 1.001, ···, 1.009; the logarithms of these 35 factors are permanently incorporated in the machine. The logarithm of the remaining factor is computed by using six terms of the logarithmic series. Antilogarithms are found by a similar process, using the exponential series. Sines (or cosines) are found directly from 11 terms of the two well-known series, with x limited to $\frac{1}{4}\pi$ and the machine choosing the appropriate series and supplying signs. Each of these last three processes takes about a minute, as compared with six seconds for multiplication or twelve for division or one third of a second for an addition or subtraction. Thus all elementary logarithmic, circular, exponential and hyperbolic functions can be generated by the machine itself without reference to any extraneous tables.

The brain of the machine is the sequence control tape, whose width is divided into three sections, in each of which coded instructions are punched. Section A directs the machine where to find a number that is to be operated on, be it a constant, or in a storage register, or on an auxiliary tape or on a Hollerith card. Section B directs where this number or a new one resulting from an operation involving it is to go: to another storage register, to the multiplying, dividing, logarithm, exponential or sine components of the "mill," to a new Hollerith card, or to an electromatic typewriter that types the output of the machine. Section C dictates the process that is to be performed. With these "prefabricated" instructions and its data, the machine runs continuously until its problem is completed. If the same process is to be repeated over and over again, as in table-making, an endless tape is used. If the tape is one for a standard process like quadrature, which is independent of the actual values used, it is preserved in the tape library and may be used every time this process is called for.

The typing of its results is an admirable feature of the machine. Tables, with the appropriate intercolumnar and

interlinear spaces, can be reproduced cheaply and accurately by photolithography, without the drudgery and risk of error associated with copy-making, printers' composition and proof-reading. Aiken estimates that by running the machine continuously it will do six months' work of a well-equipped manual computer in 24 hours.

In view of the lack of any extensive published bibliography of numerical analysis, the 65-page bibliography is particularly appreciated. *L. J. Comrie* (London).

Aiken, Howard H., and Hopper, Grace M. The automatic sequence controlled calculator. I. Elec. Engrg. **65**, 384–391 (1946).

Aiken, Howard H., and Hopper, Grace M. The automatic sequence controlled calculator. II. Elec. Engrg. **65**, 449–454 (1946).

Aiken, Howard H., and Hopper, Grace M. The automatic sequence controlled calculator. III. Elec. Engrg. **65**, 522–528 (1946).

For a more detailed account cf. the book reviewed above.

Laderman, Jack, and Abramowitz, Milton. Application of machines to differencing of tables. J. Amer. Statist. Assoc. **41**, 233–237 (1946). [MF 16772]

Highly specialized machines designed for scientific computations are not widely available. The authors discuss the adaptability of I.B.M. punched card equipment, the Underwood-Elliott Fisher Sundstrand accounting machine, as well as standard makes of calculating machines, to the problems of scientific computation, with particular reference to constructing tables of differences and to tabulating functions by means of differences. *W. E. Milne* (Corvallis, Ore.).

Womersley, J. R. Scientific computing in Great Britain. Math. Tables and Other Aids to Computation **2**, 110–117 (1946).

***Tables of Fractional Powers.** Prepared by the Mathematical Tables Project, conducted under the sponsorship of the National Bureau of Standards. Lyman J. Briggs, Director, National Bureau of Standards; Arnold N. Lowan, Director, Mathematical Tables Project. Columbia University Press, New York, 1946. xxx+489 pp. $7.50.

The tables are to 15 decimals, except for the last one. The first part tabulates A^x for the following ranges: (1) $A=2(1)9, x=.001(.001).01(.01)1$; (2) $A=10, x=.001(.001)1$; (3) $A=\pi$, $x=.001(.001)1$, with a supplementary table for $\pm x=\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1(1)12$; (4) $A=.01(.01)1$, $x=.001(.001)-.01(.01)1$; (5) $A=10^{-3}P$, P a prime between 100 and 1000, $x=.001(.001).01(.01)1$. The second part tabulates x^a for $a=\pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm\frac{1}{4}, \pm\frac{3}{4}$, $x=0(.01)10$; and, finally (to 7 decimals) $a=.01(.01)1$, $x=0(.001)$ various; $0(.01)1$. The tables are reproduced photographically from typescript. There is a foreword by F. Bernstein discussing the type of problems whose solution is facilitated by the use of the tables and giving an example. There is a bibliography (76 items) of tables of nonintegral powers and powers of nonintegral numbers. *R. P. Boas, Jr.* (Providence, R. I.).

Cerrillo, Manuel, and Suarez Diaz, Jorge. Functions associated with electrical transmission lines. Comisión Impulsora y Coordinadora de la Investigación Científica. (Mexico). Anuario **1944**, 67–94 (1945). (Spanish)

When dimensionless quantities are introduced, the propagation constant and other characteristic constants of a transmission line are shown to depend on a small number of standard functions. With the usual meanings of R, L, C, G, ω, the author uses the dimensionless parameters $x=\omega(LC/RG)^{\frac{1}{2}}$, $m=(LG/RC)^{\frac{1}{2}}$, $p=\frac{1}{2}(m^2+m^{-2})$, and shows that the propagation constant depends on A and B, defined by

$$2^{-\frac{1}{2}}\{(x^4+2px^2+1)^{\frac{1}{2}}\pm(1-x^2)\}^{\frac{1}{2}},$$

the natural impedance on

$$C=(mA+xB)/(m^2+x^2), \quad D=(mB-xA)/(m^2+x^2)$$

and the propagation constant of the "inverse line" on

$$E=A(x^4+2px^2+1)^{-\frac{1}{2}}, \quad F=B(x^4+2px^2+1)^{-\frac{1}{2}}.$$

He gives tables to six decimals of $A, \cdots, F$ for $p=1(1)10$ and $x=0(0.1)1(0.2)2(0.25)3(0.5)4(1)10(5)20$. There are also corresponding tables for the case $G=0$ when $x=\omega(L/R)^{\frac{1}{2}}$. The tabulated functions are also shown in graphs (plotted against x, with p as parameter). *A. Erdélyi.*

Numerical tables for the calculation of radiators. Comisión Impulsora y Coordinadora de la Investigación Científica. (Mexico). Anuario **1944**, 125–247 (1945). (Spanish)

These tables, for use in calculations with electromagnetic radiators, give, in effect, for argument r (although not so symbolised in the tables), 5-decimal values of $\pi r(1-\cos\theta)$ for $r=0(0.1)5$ and $\theta=0(1^\circ)180^\circ$.

Confidence in the table is shaken by the discovery that the preliminary 6-decimal table of πr is based on $\pi=3.14159$ and so has a progressive error in its 50 values rising at $r=5$ to 13 units of the last decimal; these erroneous values have been used in part of the main table, but not all. The table for each θ simply gives 50 multiples of a constant; in these days of calculating machines it would seem that one page of 180 constants would have sufficed, rather than 90 pages of mere multiples.

A supplementary table gives 9-figure values of $\sin x$ and $\cos x$, taken from the New York W.P.A. (now M.T.P.) tables [Tables of Sines and Cosines for Radian Arguments, 1940; these Rev. **2**, 64], together with 8-figure values of $x^{-1}\sin x$ and $x^{-1}\cos x$ for $x=0(0.001)1$. *L. J. Comrie.*

***Tables of the Modified Hankel Functions of Order One-Third and of their Derivatives, by the Staff of the Computation Laboratory.** The Annals of the Computation Laboratory of Harvard University, vol. 2. Harvard University Press, Cambridge, Mass., 1945. xxxvi+235 pp. $10.00.

The functions in question are

$$f(z)=T^{\frac{1}{3}}J_{-\frac{1}{3}}(T), \quad g(z)=T^{\frac{1}{3}}J_{\frac{1}{3}}(T), \quad h_s(z)=T^{\frac{1}{3}}H_{\frac{1}{3}}^{(s)}(T),$$

where $T=\frac{2}{3}z^{\frac{3}{2}}$ and $s=1, 2$. They satisfy the differential equation $u''+zu=0$, and occur in various problems of mathematical physics (for example, investigations on the intensity of light in the neighbourhood of a caustic, the theory of diffraction and refraction of waves, buckling of vertical posts under their own weight) as well as in asymptotic solutions of linear differential equations of the second order.

The introduction gives the definition of the functions, their relations with each other and with other functions, notably the Airy integrals, integral representations, asymptotic expansions, details of the method of computation, notes on interpolation and subtabulation, and a bibliography of 41 items.

The principal tables give the real and imaginary parts of $h_1(x+iy)$, $h_2(x+iy)$ and of their derivatives to eight decimals for $|x+iy|\leqq 6$, the interval both in x and y being 0.1.

Supplementary tables give the coefficients in the power-series of $f(z)$, $g(z)$ and of their derivatives; the coefficients in the asymptotic expansions of $h_1(z)$ and $h_2(z)$; some constants useful in the work with these functions; 8 decimal values of the real and imaginary parts and of the moduli of the roots of the equation $h_2(z)=0$ together with the values at these roots of $h_1(z)$, $h_1'(z)$, $h_2'(z)$, and of the roots of $h_2'(z)=0$ together with the values at these roots of $h_1(z)$, $h_1'(z)$, and $h_2(z)$.

This useful volume is accompanied by some figures among which the following may be mentioned (we put $h_1(x+iy)=X+iY$): contours in the (x, y)-plane for $X=$constant and $Y=$constant, and in the (X, Y)-plane for $x=0$ and $y=0$; and graphs of X and Y along the real axis $y=0$, along the imaginary axis $x=0$, and along the line $y=3^{\frac{1}{2}}x$ (where $X=0$). *A. Erdélyi* (Edinburgh).

Dwight, H. B. Table of the Bessel functions and derivatives J_2, J_1', J_2', N_2, N_1', N_2'. J. Math. Phys. Mass. Inst. Tech. **25**, 93–95 (1946).

These tables are calculated from $x=0.00$ to $x=0.20$ at intervals of 0.01, from $x=0.2$ to $x=10.0$ at intervals of 0.1. They give J_2, J_1' and J_2' mainly to 8 places, N_2, N_1' and N_2' to 6 places. The function $N_n(x)$ is the same as $Y_n(x)$ of Mathematical Tables, vol. 6 [British Association for the Advancement of Science, Cambridge University Press, 1937], namely

$$N_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{n+2r}}{r!(n+r)!}\left\{2\log(x/2)+2C-\sum_{m=1}^{n+r}m^{-1}-\sum_{m=1}^{r}m^{-1}\right\}-\sum_{r=0}^{n-1}(x/2)^{-n+2r}(n-r-1)!/r!.$$

S. C. van Veen (Delft).

More zeros of certain Bessel functions of fractional order. Math. Tables and Other Aids to Computation **2**, 118–119 (1946).

The following table is a continuation of the one given by M. Abramowitz [same journal **1**, 353–354 (1945); these Rev. **6**, 132]. The quantities $j_{\nu,s}$, where $j_{\nu,s}$ denotes the sth zero of $J_\nu(x)$, are given for $\nu=\pm\frac{1}{4}$, $\pm\frac{1}{2}$, $\pm\frac{3}{4}$, $\pm\frac{3}{4}$, and for $s=$[8(1)30; 10D]. These values are believed to be correct to well within a unit in the tenth decimal. Thus this table supplements the one given by Abramowitz up to $s=7$ or 8. The roots $j_{\nu,s}$ were calculated from the well-known asymptotic expansion. [The table was computed by the New York Mathematical Tables Project.] *Extract from the paper.*

Blanch, Gertrude. On the computation of Mathieu functions. J. Math. Phys. Mass. Inst. Tech. **25**, 1–20 (1946).

For the Mathieu equation $y''+(a-2q\cos 2x)y=0$, it is well known that certain values of a, described as characteristic values, lead to periodic solutions. Ince, Goldstein and, more recently, McLachlan [Philos. Mag. (7) **36**, 403–414 (1945); these Rev. **7**, 486] have developed methods of obtaining these characteristic values from a continued fraction. The author remarks, "there does not seem to appear in the literature any method for improving the accuracy of the characteristic values, except by cumbersome iteration." She then develops a method which corrects not only an approximate characteristic value, but also the coefficients in the series for the periodic solutions.

In the two illustrations the initial approximations are good to 12 significant figures, and the final to 21 and 17 figures, respectively. It is not clear to the reviewer why 12-figure values are not ample, considering that in practical applications three, four or five figures of the Mathieu function suffice; nor is he convinced, from his own experience of computing characteristic values for fractional orders, that iteration is "cumbersome." *L. J. Comrie* (London).

Bodewig, E. On Graeffe's method for solving algebraic equations. Quart. Appl. Math. **4**, 177–190 (1946). [MF 16963]

The author discusses Graeffe's root squaring method for solving algebraic equations, with emphasis on cases where there are multiple roots or roots with equal absolute values. He concludes that the method is superior to others because (a) it gives approximations to all the roots simultaneously, (b) it gives the complex as well as the real roots, (c) it converges rapidly (quadratic convergence), (d) it requires no first approximation to begin with.

The process of successive squaring of roots will split the given equation Q into several equations of lower degree M_i, one for each different modulus of the roots. The modulus R_i of the roots corresponding to the equation M_i is found from the ratio of two of the coefficients in M_i. The equation M_i in the variable X is then transformed by the relation $X=R_iY$ into an equation in $Y=e^{i\phi}$, where ϕ is the argument of the roots. The resulting equation is reciprocal, that is, Y and $1/Y$ are solutions concurrently. After dividing out any factors $Y-1$ that occur, the substitution $Z=Y+1/Y$ reduces the equation to one with real roots, which can be solved by the usual Graeffe method. The procedure yields many extraneous values from which the true solution must be separated by substitution. *P. W. Ketchum.*

Rademacher, H. A., and Schoenberg, I. J. An iteration method for calculation with Laurent series. Quart. Appl. Math. **4**, 142–159 (1946). [MF 16960]

The problem is to solve an algebraic equation in which the coefficients $a_n(z)$ are analytic functions of a parameter z and have known Laurent expansions in a ring about the origin. It is desired to find the numerical values of the coefficients of the Laurent expansions of the solutions. A special case is the problem of reciprocation of Laurent series. An obvious transformation puts the unit circle in the ring. A first approximation to the solution is found by interpolating a Laurent series at the mth roots of unity; this amounts to ordinary trigonometric interpolation. This approximation is improved by successive application of a modification of Newton's well-known algorithm for the roots of an equation, a modification previously used by H. Schwerdtfeger [J. S. Frame, Amer. Math. Monthly **52**, 212–214 (1945)]. By an extension of a theorem of Hotelling and Lonseth [Hotelling, Ann. Math. Statistics **14**, 440–441 (1943); these Rev. **5**, 245] on the reciprocation of matrices, it follows that the successive approximations converge if the first approximation is sufficiently close to the exact solution. Here "sufficiently close" is measured by the norm $N=\sum|c_\nu|$, where the c_ν are the coefficients of the Laurent series. The authors prove that the first approximation will satisfy the last condition if m is sufficiently large. The same results apply for the case where the coefficients $a_n(z)$ are not analytic in a ring but have absolutely convergent Fourier expansions on the unit circle.

As a numerical example the Laurent expansion of the reciprocal of the Bessel function $J_0(z)$ in the ring between the first two positive zeros of $J_0(z)$ is calculated.

P. W. Ketchum (Urbana, Ill.).

Salzer, Herbert E. Note on a formula for the solution of an arbitrary analytic equation. Quart. Appl. Math. **4**, 306–307 (1946).

In connection with a paper by Blaskett and Schwerdtfeger [same Quart. **3**, 266–268 (1945); these Rev. **7**, 218], the author calls attention to a paper by Van Orstrand [Philos. Mag. (6) **19**, 366–376 (1910)].

Mayantz, L. S. Perturbation method with the application of double iteration. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 313–316 (1945). [MF 16658]

The paper gives a method of successive approximations for finding the characteristic vectors and characteristic values of a linear transformation W in n-dimensional space, starting from known characteristic vectors and values of a neighboring transformation W_0. The equation $W'r=\lambda r-W_0r$ for the characteristic vectors r and the characteristic values λ of the perturbation operator $W'=W-W_0$ is expanded in the usual manner into a system of n linear equations in the components of r, assuming the characteristic vectors of W_0 as the space basis. These n homogeneous equations are made nonhomogeneous by dividing through by one of the unknown components r_i. The ith equation of the resulting set is then separated from the others and solved for the unknown parameter λ. An iteration procedure is obtained on substituting a given set of ratios r_j/r_i into this equation to give an approximation to λ and then substituting this value of λ into the remaining $n-1$ equations to give, when solved, a new set of ratios r_j/r_i, etc. These $n-1$ equations may in turn be solved by iteration in the usual way. Conditions for convergence and estimates of error of the resulting double iteration procedure are given. *P. W. Ketchum.*

Thurstone, L. L. A single plane method of rotation. Psychometrika **11**, 71–79 (1946). [MF 16897]

A method of computation involving graphical, numerical and empirical elements for the rotation process in factor analysis. A single sequence of iterations is required for each hyperplane sought. *J. W. Tukey* (Princeton, N. J.).

***Nørlund, N. E. Determination of the weights for the unknowns in graduation of elements.** Festskrift til Professor, Dr. Phil. J. F. Steffensen fra Kolleger og Elever paa hans 70 Aars Fødselsdag 28. Februar 1943, pp. 126–128. Den Danske Aktuarforening, Copenhagen, 1943. (Danish)

When a system of linear equations is graduated by the method of least squares the weights of the graduated quantities may be expressed explicitly in terms of the coefficients of the linear equations. *P. Johansen* (Copenhagen).

Aitken, A. C. Studies in practical mathematics. IV. On linear approximation by least squares. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 138–146 (1945). [MF 16292]

Starting from the moment-matrix of Frisch (called by the author the "variance matrix") the author develops the Gaussian method in a concise modern form by the use of vectors and matrices. The second and main feature of the paper is the introduction of the "graduating matrix" G, which in the case of equal weights is $G=P(P'P)^{-1}P'$, where P is given by $Pa=u$ and a is the unknown vector. This G can be constructed once for all when the number of observations and the degree of the approximating polynomial are given. Then the values of the curve fitting the observations will be given by $y=Gu$. *E. Bodewig* (The Hague).

Banachiewicz, T. On the accuracy of least squares solution. Ark. Mat. Astr. Fys. **31B**, no. 8, 3 pp. (1945).

In contrast to the classical theory where the errors are determined by the weights, the author introduces the notion of faultiness F of the unknowns: $F=\mu/\mu_0$, where μ is the mean error, μ_0 the standard mean error. To calculate F_i for the ith unknown in a system of normal equations, compute $\mathfrak{B}=1/\sqrt{\mathfrak{A}}$, where $\mathfrak{A}$ is the Cracovian of the coefficients of the equations. Then, if the b_{ik} are the elements of the ith column of $\mathfrak{B}$, $F_i^2=b_{i1}^2+\cdots+b_{in}^2$. The difficulty consists, of course, in finding $\sqrt{\mathfrak{A}}$. *E. Bodewig* (The Hague).

Romanovsky, W. Sur certains théorèmes concernant la méthode des moindres carrés. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 263–265 (1946).

Soient $x_1, \cdots, x_n$ des variables aléatoires normales et indépendantes ayant pour moyennes $a_i=\sum_{k=1}^{r}a_{ik}\alpha_k$, $i=1, \cdots, n$, $r<n$, et pour dispersions $\sigma_i=\sigma p_i^{-\frac{1}{2}}$, où les $p_i>0$ sont connus, σ est inconnu et la matrice des a_{ik} a le rang r, le plus grand possible. Soit S_0 le minimum de $S=\sum_i p_i(x_i-a_i)^2$ avec $\alpha_k=\alpha_k^0$. Soit $\theta=c_1\alpha_1+\cdots+c_r\alpha_r$ et $T=c_1\alpha_1^0+\cdots+c_r\alpha_r^0$, où les c sont arbitraires. Alors l'auteur indique (1) que les quantités $\chi_1=S_0^{\frac{1}{2}}/\sigma$ et $\chi_2=\sigma^{-1}|T-\theta|(-m\Delta/\Delta_e)^{\frac{1}{2}}$, où $m=n-r$, possèdent les répartitions indépendantes χ^2,

$$P(\chi_1, m)=\chi_1^m e^{-\frac{1}{2}\chi_1^2}/2^{(m-2)/2}\Gamma(m/2), \quad P(\chi_2, 1)=\pi^{-\frac{1}{2}}e^{-\frac{1}{2}\chi_2^2};$$

(2) que la quantité $t=S_0^{-\frac{1}{2}}T-\theta(-m\Delta/\Delta_e)^{\frac{1}{2}}$ a la distribution de "Student,"

$$S(t, m)=C(1+t^2/m)^{-(m+1)/2}, \quad C=\Gamma((m+1)/2)/\Gamma(m/2)(m\pi)^{\frac{1}{2}}.$$

Ici Δ désigne la déterminante des A_{kl} et Δ_e désigne Δ augmentée à la droite avec la colonne $c_1, \cdots, c_r, 0$ et au bas avec la règle $c_1, \cdots, c_r, 0$; $A_{kl}=\sum_i p_i a_{ik}a_{il}$. *E. Bodewig.*

Mineur, Henri. Sur la meilleure représentation d'une variable aléatoire par fonction linéaire de variables à choisir parmi des variables aléatoires données. Ann. Astrophysique **7**, 17–30 (1944).

Let the $\mathbf{x}_k$ ($k=1, \cdots, n$) be dependent random variables and let it be desired to express ξ as a linear combination of the $\mathbf{x}_k$ to fit certain observations on the $n+1$ variables. The author points out that frequently the use of all the $\mathbf{x}_k$, as implied in the method of least squares, is not only unnecessary, but even misleading. He indicates two methods for a practical decision of how many and which of the $\mathbf{x}_k$ should be used. The first, called method of partial quadratic residues, approximates ξ by each of the $\mathbf{x}_k$ separately and selects, to begin with, the particular $\mathbf{x}_k$ with the smallest residual effect. This residuum is then treated in a similar way, and one proceeds until a stage arrives where the successive residua no longer decrease appreciably. The second method uses partial correlation instead. The two methods are shown to be essentially equivalent.

W. Feller (Ithaca, N. Y.).

Mineur, Henri. Extension de la méthode des moindres carrés. Application à la détermination de l'apex au moyen des mouvements propres. Ann. Astrophysique **7**, 121–132 (1944).

L'auteur donne une nouvelle méthode des moindres carrés, qui consiste à chercher la loi de probabilité des inconnues

x, y, z du système $a_i x+b_i y+c_i z+R_i=\xi_i$, $i=1, \cdots, N$, où les a_i, b_i, c_i, ξ_i sont connus et les R_i sont des variables aléatoires normales de valeurs moyennes 0, avec des paramètres connus h_i. Il trouve que les x, y, z sont distribués normalement et donne les valeurs moyennes x_0, y_0, z_0, qui sont définies par les équations linéaires normales, et $\sigma_x, \cdots, r_{yz}, \cdots$. Ensuite il montre comment on passe de la nouvelle méthode à l'ancienne. Dans la dernière on a $h_i=p_i h$, où les poids p_i sont connus, tandis que l'inconnue h sera déterminée.

Alors il étudie le système $a_i x+b_i y+c_i z+R_i+S_i=\xi_i$, où les S_i et R_i sont des variables aléatoires, les premières avec les paramètres connus k_i, tandis que les paramètres h_i des R_i sont connus seulement à un facteur près: $h_i=p_i h$. Ce problème mixte est résolu approximativement dans le cas où l'on connaît une valeur approximative h_0 de h: $h^{-1}=(N-3)^{-1}\sum_{i=1}^{N} P_i \rho_i^2$, où $P_i=p_i k_i/(k_i+p_i h_0)$, et les ρ_i sont les résidus: $\rho_i=\xi_i-a_i x_0-b_i y_0-c_i z_0$.

L'application de cette méthode à la détermination de l'apex au moyen des mouvements propres donne et l'erreur de détermination du mouvement propre et le mouvement résiduel de l'étoile. Les méthodes anciennes supposaient nulle l'une de ces variables. Enfin la méthode est appliquée à la détermination du zéro d'un système de magnitudes absolues. *E. Bodewig* (La Haye).

Mineur, Henri. Nouvelle méthode de lissage et méthode pour déterminer avec précision la période d'un phénomène. Ann. Astrophysique **6** (1943), 136–158 (1944).

L'auteur propose une méthode de lissage qui (1) n'est pas basée sur une forme analytique a priori, (2) fait dépender chaque valeur lissée des valeurs données voisines, (3) est applicable au cas où les valeurs données de la variable ne forment pas une progression arithmétique. Pour cela des paraboles sont menées par chaque triple de points (x_m, y_m) consécutifs donnés et les constantes des paraboles sont calculées par la méthode des moindres carrés, de sorte que, si $e-\frac{1}{2} \leqq x_m \leqq e+\frac{1}{2}$ (où e est entier) et $\xi=x_m-e$, on a les équations pour les valeurs Y_p lissées (ayant les arguments p entiers):

$$\tfrac{1}{2}(-\xi+\xi^2)Y_{p-1}+(1-\xi^2)Y_p+\tfrac{1}{2}(\xi+\xi^2)Y_{p+1}=y_m.$$

La résolution de ces équations par la méthode des moindres carrés se trouve beaucoup simplifiée parce que dans chaque équation il n'y a que trois coefficients différents de zéro. La méthode proposée permet de déterminer la meilleure valeur de la période d'un phénomène périodique.

E. Bodewig (La Haye).

McShane, E. J. An interpolation formula. Amer. Math. Monthly **53**, 259–264 (1946). [MF 16450]

A general expression is developed for an arbitrary polynomial in terms of its values for certain equally spaced values of the independent variable and certain finite differences of these values. All the conventional nonosculatory finite difference interpolation formulae for equally spaced arguments are obtained from this general expression by assigning particular values to parameters in the general expression.

The wording of the first two sentences of the paper seems to ignore the existence of the large body of literature on osculatory interpolation. See, for example, bibliography given by the reviewer [Trans. Actuar. Soc. America **45**, 202–265 (1944), in particular, pp. 259–260; these Rev. **7**, 84]. *T. N. E. Greville* (Washington, D. C.).

Ghizzetti, Aldo. Su un nuovo procedimento di interpolazione. Ricerca Sci. **16**, 78–80 (1946).

An interpolating function whose curve passes through $n+1$ points determined by a given function may sometimes display an undesirably large total variation in the interval of interpolation as compared with the total variation of a polygonal line with vertices at the given points. The author proposes a general procedure for improving this situation. The idea is to use an interpolating function containing $n+p+1$ undetermined coefficients, that is, p more than are needed to make the curve go through the given points. The coefficients are then determined so as to minimize the integral of the square of the derivative of the interpolating function, subject to the $n+1$ conditions imposed by passage through the given points. In general, this leads to a problem of Lagrange in the calculus of variations and is solved by the use of Lagrange's multipliers. For the case of polynomial interpolation, however, the work can be arranged so as to yield a simple problem in the calculus of variations. The theory is illustrated by a numerical example.

W. E. Milne (Corvallis, Ore.).

Schoenberg, I. J. Contributions to the problem of approximation of equidistant data by analytic functions. Part B. On the problem of osculatory interpolation. A second class of analytic approximation formulae. Quart. Appl. Math. **4**, 112–141 (1946). [MF 16959]

An interpolation formula is characterized as of type D^m, C^s, E^r, s, if the even function $L(x)$ used in defining it [see part A, same Quart. **4**, 45–99 (1946); these Rev. **7**, 487] is a composite polynomial function of degree m, if the order of contact is μ, if r is the highest degree of polynomials for which the interpolation formula is exact and if s is the "span": that is, the maximum number of given values y_n required to obtain a single interpolated value $F(x)$. Furthermore, an "ordinary" interpolation formula is defined as one having the property $F(n)=y_n$, while a "smoothing" interpolation formula is defined as one which reduces, for integral x, to a smoothing formula as defined in part A. The author develops a class of ordinary interpolation formulae of type D^k, C^{k-2}, E^{k-1}, $s=2k-2$ (k even), $s=2k-1$ (k odd), for each positive integer k; a class of smoothing interpolation formulae of type D^{k-1}, C^{k-2}, $E^{\min(2m-1,\,k-1)}$, $s=k+2m-2$, for all integers m and k such that $0 \leqq 2m-2<k$; and a further class of smoothing interpolation formulae of type D^{k-1}, C^{k-2m}, E^{2m-1}, $s=k$, for all integers m and k such that $0 \leqq 2m \leqq k-1$. The formulae of the second class mentioned are given by $L(x)=\sum_{n=0}^{m-1}(-1)^n d_{2n}^{(k)} \delta^{2n} M_k(x)$, where $\delta f(x)$ denotes $f(x+\frac{1}{2})-f(x-\frac{1}{2})$,

$$M_k(x)=(2\pi)^{-1}\int_{-\infty}^{\infty}(2u^{-1}\sin\tfrac{1}{2}u)^k e^{iux}du,$$

and the coefficients $d_{2n}^{(k)}$ are given by: $(2v^{-1}\operatorname{arc\,sin}\frac{1}{2}v)^k=\sum_{n=0}^{\infty} d_{2n}^{(k)} v^{2n}$. The expressions for the formulae of the other two classes are too complicated to reproduce here. Finally, a class of analytic interpolation formulae is developed whose basic function $L(x)$ dampens more rapidly than in the case of the ordinary analytic interpolation formulae given in part A. While the ones given in part B are not ordinary formulae, no general proof is given that they are actually "smoothing" formulae in the strict sense of the author's definition. Tables are given to facilitate their application.

T. N. E. Greville (Washington, D. C.).

Williams, J. D. An approximation to the probability integral. Ann. Math. Statistics **17**, 363–365 (1946).

It is shown that

$$(2\pi)^{-\frac{1}{2}}\int_{-\sigma}^{\sigma} e^{-t^2/2}dt \leqq \{1-e^{-2\sigma^2/\pi^2}\}^{\frac{1}{2}}$$

and that the error is never greater than three-fourths of one per cent. The inequality follows from a simple comparison of the total probability mass in a circle about the origin with that in a concentric square of same area. The estimate of the error is seen from a tabulation giving the values for $x=[0(.1)2]$. Even better elementary approximations are suggested, but they lose the advantage of the one-sided inequality. *W. Feller* (Ithaca, N. Y.).

Laurila, E. Ein Produktintegraph. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **29**, 12 pp. (1945). [MF 16477]

The author gives a detailed description of a new mechanical integrator. Its main function is to draw on a "result table" a graph of the integral $y=F(X)=\int^X f(x)g(x)dx$, where the two integrand functions $f(x)$ and $g(x)$ are conveyed in graphical form on two separate "input tables" through two hand-operated curve-followers. An automatic curve-follower is provided for $f(x)$ in the form of a pliable steel band, provided $f(x)$ is smooth and $|f'(x)|<\tan 65°$ throughout. By an adjustable coupling of the movement of the input tables the two x-scales of $g(x)$ and $f(x)$ may have any ratio between 0.25 and 1.125. As all three tables are moved in the direction of the x-axis by a motor (with adjustable speed), the two "input lengths" f and g are combined to a length W proportional to $f\cdot g$ by a "similar triangle" assembly of "rails" and rotating rods. The integration of W is with the help of a fixed sharp-edged "rolling wheel" whose rail is guiding the boss of the result table in the direction $\tan^{-1} W$ against the x-axis, so that its pencil draws the integral $y=\int Wdx=\int fgdx$.

A standard application is the plotting of incomplete moments $\int^X g(x)x^n dx$ of a function g, with standard input steel bands x^n. By coupling input and result tables solutions of both $y'+p(x)y=g(x)$ and $y'+(y+f(x))g(y)=0$ can be drawn. No statement on accuracy is made. The integrator was built in Finland; its present location is not stated. *H. O. Hartley* (London).

Mikeladze, Sh. On numerical integration. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 166–167 (1945). [MF 16405]

The author derives formulas for expressing the integral

$$\int_{a-mh}^{a+mh} f(x)dx$$

as a linear combination of values of the function $f(x)$ and of its derivative taken at $2r+1$ symmetrically located points $a, a\pm t_1h, a\pm t_2h, \cdots, a\pm t_rh$. The general formula is exact if $f(x)$ is a polynomial of degree less than $4r+2$. Several misprints were noted. *W. E. Milne* (Corvallis, Ore.).

***Odgaard, Helge. The remainder term in some quadrature formulas.** Festskrift til Professor, Dr. Phil. J. F. Steffensen fra Kolleger og Elever paa hans 70 Aars Fødselsdag 28. Februar 1943, pp. 129–132. Den Danske Aktuarforening, Copenhagen, 1943. (Danish)

The author studies the quadrature formula for $\int_{-n}^{m} f(x)dx$ obtained by integrating the Lagrange interpolation polynomial which uses the $2n$ points $\pm a_1, \cdots, \pm a_n$. Put $P(x)=(x^2-a_1^2)\cdots(x^2-a_n^2)$. It is shown that if

$$\int_{-m}^{x} P(t)(t-a_n)^{-1}dt$$

does not change sign in $(-m, 0)$, then the remainder is of the form

$$(2n!)^{-1}f^{(2n)}(\xi)\int_{-m}^{m} P(t)dt,$$

where $|\xi|\leqq\max(a_i, m)$. The condition is satisfied in the so-called MacLaurin case $a_i=i-\frac{1}{2}$, $m=n$. This case has been studied by A. Walther [Skand. Aktuarietidskr. **8**, 148–162 (1925)] who obtained the same result for formulas using an odd number of points. *W. Feller* (Ithaca, N. Y.).

Born, Max, Fürth, R., and Pringle, R. W. A photoelectric Fourier transformer. Nature **156**, 756–757 (1945). [MF 15121]

A very brief description is given of a photoelectric device for producing and displaying on a cathode-ray oscilloscope screen a graph of the function

$$g'(y)=\int_a^b f(x)\cos(yx+\delta)dx.$$

Several photographs show the impressive results obtained in illustrative examples. The amount of description is not adequate for a review of the device itself, but the authors indicate that a detailed description will be published elsewhere. *S. H. Caldwell* (Cambridge, Mass.).

Hägg, G., and Laurent, T. A machine for the summation of Fourier series. J. Sci. Instruments **23**, 155–158 (1946).

The widespread application of the Beevers and Lipson strip method for Fourier synthesis in the analysis of crystal structure has stimulated a number of workers to develop equipment for mechanizing the routine application of the process. The authors describe a device for this purpose which appears to be well conceived and executed from the standpoints of reliability, operation, accuracy and freedom from operational complexity. With suitable modification of the input transformer, it can be adapted for direct use on any ordinary power system without the need for special sources of electric power.

The machine operates by adding, at each interval point, a set of voltages corresponding in magnitude and phase to the values, at those points, of the separate Fourier terms. These summations can be performed at intervals of 6° over the range 0 to 2π. A tapped transformer supplies a set of voltages corresponding to each 6° of a sinusoidal variation from zero to the maximum voltage of the transformer. These taps appear on both sides of a center point so that both "positive" and "negative" voltages are available. Corresponding to each harmonic order, there is a selector switch and contact ring assembly which derives the required voltage from the transformer by establishing a connection to the proper tap for that particular harmonic and for the particular angle being calculated. All of these selector switches are mounted on a common shaft so that the switching from one interval to the next is accomplished by a single crank motion. The voltage picked up by each selector switch is connected to a potentiometer which attenuates the voltage in proportion to the coefficient of each term. These attenuated voltages are finally combined in an adding network and passed to a measuring device where the sum is read directly. Provision is made for a number of direct-

reading scales so that both large and small results may be read with adequate accuracy.

In an actual sample reported, the maximum error at about mid-scale seems to be about 3%. The summation of 30 points for 13 terms was accomplished in 9 minutes.

S. H. Caldwell (Cambridge, Mass.).

Werenskiold, W. A summary method in harmonic analysis. Meteorologiske Ann. **1**, 137–147 (1942). [MF 16594]

If, in the development $t=a_0+\sum(a_k \cos kv+b_k \sin kv)$, the higher terms are insignificant, the lower terms may be calculated as follows. Let the period be divided into $4m$ equal parts; write the sequence $t_1, t_2, \cdots, t_{4m}$ as a matrix of h rows (the second row, for example, beginning with t_{4q+1}), add the columns and so obtain the sums $s_1, s_2, \cdots, s_{4q}$, where $q=m/h$. Then form the sums

$$s_1+\cdots+s_q=\mathrm{I}_h, \quad s_{q+1}+\cdots+s_{2q}=\mathrm{II}_h,$$
$$s_{2q+1}+\cdots+s_{3q}=\mathrm{III}_h, \quad s_{3q+1}+\cdots+s_{4q}=\mathrm{IV}_h,$$

and finally calculate $A_h=D_h-E_h$, $B_h=D_h+E_h$, where $D_h=\mathrm{I}_h-\mathrm{III}_h$, $E_h=\mathrm{II}_h-\mathrm{IV}_h$. Then for $q=1, 2, 3$ we have

$$a_k=C(A_k+\tfrac{1}{2}A_{2k}-\tfrac{1}{2}A_{4k}-\cdots), \quad b_k=C(B_k-\tfrac{1}{2}B_{2k}-\tfrac{1}{2}B_{4k}-\cdots),$$

where $C=(1/2k) \sin k\pi/n$. *E. Bodewig* (The Hague).

Rybner, Jørgen. Fourier analysis of frequency modulated oscillations with saw-tooth variation of instantaneous frequencies (pulse modulation). Mat. Tidsskr. B. **1946**, 97–112 (1946). (Danish) [MF 16311]

J. R. Carson [Proc. I.R.E. **10**, 57–64 (1922)] has deduced the Fourier series of the frequency modulated vibration $u=A \sin(\Omega t+(ka/\omega) \sin \omega t)$. The author replaces the "sine modulation" by a "saw tooth modulation" and gives approximate formulas for the Fourier coefficients.

František Wolf (Berkeley, Calif.).

Hirsovies, Marc. Résolution d'équations différentielles linéaires par un procédé de corrections successives. C. R. Acad. Sci. Paris **221**, 167–168 (1945). [MF 14246]

The author suggests a modification of the method of successive approximations for a linear equation

$$(1) \qquad \sum_{i=1}^{n} A_i(x)F_i(y)=B(x)$$

involving additive operators F_i applied to the unknown function $y(x)$. He defines a series of terms $y_m(x)$ by

$$\sum_{i=1}^{k} A_iF_i(y_1)=B, \quad \sum_{i=1}^{k} A_iF_i(y_{m+1})=-\sum_{i=k+1}^{n} A_iF_i(y_m), \quad m>1.$$

If $\sum_m y_m(x)$ converges it represents the solution of (1). The case $k=1$ for differential operators represents Picard's classical method. No discussion of convergence conditions is given. *H. O. Hartley* (London).

Ede, A. J. A new form of chart for determining temperatures in bodies of regular shape during heating or cooling. Philos. Mag. (7) **36**, 845–851 (1945).

Kazinsky, V. A. On the application of curvatures. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 622–625 (1945). [MF 16641]

A gravimetric survey with the aid of torsion balances theoretically allows the observation of the second derivatives T_{xy} and $T_\Delta=T_{x^2}-T_{y^2}$ of the potential T of disturbing masses. This article, based on a theoretical result of Eötvös, shows how to deduce from these observed derivatives the deflection (in radians) of the plumb-line from the geoid's vertical and gives a scheme of computation for the northward and eastward components of this deflection. [In practice the application of the second derivatives T_{xy} and T_Δ (curvature of the equipotential surfaces) is difficult and is a source of great errors since the corrections for the influence of the topography on T_{xy} and T_Δ cannot be computed correctly.] *E. Kogbetliantz* (New York, N. Y.).

Yudin, M. I. The general case of locating a point on a plane by three angle measurements. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 472–475 (1945).

Ansermet, A. Les calculs de compensation et le contrôle des poids. Schweiz. Z. Vermessgswes. Kulturtech. **43**, 176–180 (1945). [MF 14002]

L'auteur donne un nouveau contrôle dans le calcul des compensations. Soient v_i les résidus, $l_i+v_i=f_i(x_1, \cdots, x_n)$, $i=1, \cdots, n$, et les p_i les poids des quantités mesurées l_i. Après compensation les expressions l_i+v_i auront les poids P_i. Alors il existe la relation simple entre les p_i et P_i: $\sum_1^n p_i/P_i=u=$ nombre des inconnus. *E. Bodewig.*

Bachmann, W. K. Calcul symbolique des coefficients de poids. Schweiz. Z. Vermessgswes. Kulturtech. **43**, 131–139 (1945). [MF 14001]

L'auteur donne une explication du calcul symbolique des coefficients de poids, inventé par J. M. Tienstra, afin de faciliter la lecture de ses publications photogrammétriques.

E. Bodewig (La Haye).

Actuarial Mathematics

Varoli, Giuseppe. Ammortamenti in regime di capitalizzazione semplice. Period. Mat. (4) **22**, 104–119 (1942).

The amortization of a debt in the theory of simple interest is considered. It is assumed that the debt is repaid by periodical and equal installments. These may be determined by the agreement concerning the debt in various ways. Three possibilities are discussed. (1) At the moment the loan is taken the present value of all the installments should be equal to the debt. (2) At the moment the debt is paid up the amount of all the installments should be equal to the amount of the debt. (3) Each installment should be used partly to pay the interest for the residual indebtedness and partly to reduce the debt further. Each of these agreements leads to a different installment, a fact apparently conflicting with the principle of equivalence. The author points out that the principle of equivalence is nevertheless satisfied, provided a suitable function of valuation is used which takes into proper consideration the beginning of the process of amortization. *E. Lukacs.*

***Zwinggi, Ernst. Versicherungsmathematik.** Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, 1. Mathematische Reihe, Band I. Verlag Birkhäuser, Basel, 1945. 199 pp. Bound, 27 Swiss francs.

The book originated in courses on actuarial mathematics given by the author at the University of Basel. It intro-

duces the student to the most important actuarial methods. The presentation is concise and lucid, leading the reader from well stated assumptions to formulae useful in practical applications.

The text is divided into six parts. I. Bases of calculation. The starting point is Loewy's theory of intensities [S.-B. Heidelberger Akad. Wiss. Abt. A. **1917**, no. 6]. This permits one to consider not only mortality tables but also tables with several causes of decrement. Thus it is possible to develop the theory of disability insurance parallel to the theory of life insurance. II. The general insurance contract. The principle of equivalence and the definition of the premium reserve are discussed for the general case involving several causes of decrement. It is assumed throughout the text that all payments are due discontinuously. III. Single life insurance with individual premium. The usual formulae of life and invalidity insurance are derived. Net premiums as well as loadings to take care of the expenses are considered. Formulae and methods for the computation of the reserves are discussed. Several technical questions such as the division of the surplus and the determination of surrender values are treated. IV. Single life insurance with average premium. V. Joint life annuities and insurances. The second half of part V is devoted to group insurance. VI. Variations of the bases of calculation. VII. Graduation. The last two parts present a brief account of some important problems. An extensive bibliography is given, which should be very helpful to a reader who intends to penetrate deeper into the subject. *E. Lukacs* (Cincinnati, Ohio).

Zwinggi, E. Eine Produktdarstellung für das Deckungskapital. Experientia **2**, 182–183 (1946). [MF 16855]

A relation between probabilities of survival and premium reserves and premiums for the endowment insurance is derived in continuous as well as in discontinuous notation. This relation has already been used by the author in a previous paper [Mitt. Verein. Schweiz. Versich.-Math. **45**, 375–383 (1945); these Rev. **7**, 340] for deriving several approximations. *E. Lukacs* (Cincinnati, Ohio).

Malmquist, F. A mathematical theory of life insurance based on the assumption of a decreasing force of interest. Ark. Mat. Astr. Fys. **31A**, no. 24, 106 pp. (1945).

The technical rate of interest of an insurance company must always be determined with a margin of security. Starting from the principle that this margin should increase with the duration of the insurance the author makes the following assumptions. The force of interest δ_t is a non-increasing differentiable function for $t \geqq 0$ and the derivative δ_t' is negative and continuous in a positive interval $0 \leqq t < \epsilon$. If the real conditions do not show any important decline of the rate of interest the margin of security may be released in the form of profit, which is called basic profit.

Denote by V_{t+u} or $V_{t+(u)}$ the policy value determined under the assumption that the force of interest at the time $t+u+s$ is δ_s or δ_{u+s}. The element of basic profit, generated in the time element from t to $t+dt$, is then defined by $d\omega_t = V_{t+(dt)} - V_{t+dt}$. The function $d\omega_t/dt$ is called the force of basic profit; it is shown that $d\omega_t/dt = \delta_0(V_t - V_t^I)$, where V_t^I is a policy value determined with the force of interest $\delta_s^I = \delta_s - \delta_s'/\delta_s \geqq \delta_s$. Premiums, policy values and force of basic profit may be calculated if tables of three actuarial functions are available. These functions are

$$\bar{a}_{x:\overline{n}|} = \int_0^n l_{x+s} l_x^{-1} f_s \, ds, \quad \bar{\alpha}_{x:\overline{n}|} = -\delta_0^{-1} \int_0^n l_{x+s} l_x^{-1} f_s' \, ds,$$

$$\beta_{x:\overline{n}|} = (\delta_0^2 - \delta_0')^{-1} \int_0^n l_{x+s} l_x^{-1} f_s'' \, ds.$$

In these formulae l_x denotes the number living and $f_s = \exp\{-\int_0^s \delta_t dt\}$ the function of discount.

The author then specializes the function f_s in order to simplify the computations. It is assumed that $f_s = e^{-\delta_1 s} P_s$ for $s \leqq v$ and $f_s = e^{-\delta_1 s} P_v$ for $s > v$, where $P_s = 1 - cs + \frac{1}{2}cs^2/v$ and δ, c, $v < 1/c$ are positive constants. It is shown that in this case all calculations can be performed by means of the usual actuarial tables based on the constant force of interest and tables of the values of P_t and P_t' for $t = 0, 1, \cdots, v$. Another simplification is obtained by assuming that the function of discount f_t is the weighted mean between the functions of discount corresponding to constant forces of interest σ_1 and σ_2. In this case all calculations of premiums, policy values and force of basic profit can be carried out if the usual tables based on both constant forces σ_1 and σ_2 are available. Finally, the author compares his results with the traditional theory based on a constant rate of interest. The distribution of basic profit as well as group methods of valuation are briefly discussed. *E. Lukacs.*

Jecklin, Heinrich. Näherungswerte für die gemischte Versicherung mehrerer verbundener Leben. Mitt. Verein. Schweiz. Versich.-Math. **46**, 111–126 (1946). [MF 16689]

Makeham's approximation formula of joint life premiums $P_{xy\overline{n}|} \sim P_{x\overline{n}|} + P_{y\overline{n}|} - P_{\overline{n}|}$ may be generalized in different ways in the case of more than two joint lives. One finds, for example, a recursion formula expressing the premium of n joint lives in terms of the premiums of 0, 1, $\cdots$, $(n-1)$ joint lives. A simpler but less exact approximation is

$$P_{x,x,\cdots,(k)\overline{n}|} \sim kP_{x\overline{n}|} - (k-1)P_{\overline{n}|}.$$

Similar expressions are given for approximation of the mathematical values. *P. Johansen* (Copenhagen).

***Lublin, Mogens. The law of uniform seniority and Makeham's formula.** Festskrift til Professor, Dr. Phil. J. F. Steffensen fra Kolleger og Elever paa hans 70 Aars Fødselsdag 28. Februar 1943, pp. 100–108. Den Danske Aktuarforening, Copenhagen, 1943. (Danish)

If a group of persons $(x_1, x_2, x_3, \cdots)$ can always be replaced by another group of equally aged persons, then the author proves that the forces of mortality must follow Makeham's law $\mu_x^i = a_i + b_i c^{x_i}$ with the same value of c, when certain trivial constant and linear cases are excluded. *P. Johansen* (Copenhagen).

Cattaneo, Paolo. Sulle tavole di sopravvivenza. Period. Mat. (4) **22**, 48–51 (1942).

Let l_x be the number living in a mortality table. Two formulae of approximation, expressing the force of mortality in terms of four consecutive values of l_x, are derived by fitting a parabola of the third order to the l_x.

E. Lukacs (Cincinnati, Ohio).

ASTRONOMY

Brouwer, Dirk. A survey of the dynamics of close binary systems. Astr. J. **52**, 57–63 (1946).

This investigation deals with the motions in a system consisting of two rigid spheroids of comparable masses, whose axes of rotation are not perpendicular to the plane of the orbit. The Lagrangian equations of motion are set up and their solution sought by successive approximations. In choosing the method, advantage is taken of the fact that, to a first approximation, the motion in the orbital plane and the motion of this plane may be treated independently. The resulting system of equations is similar to that of the secular variations of the inclinations and nodes in a system of the sun and two planets, which has been treated by C. V. L. Charlier [Die Mechanik des Himmels, v. 1, Leipzig, 1902, p. 358] by a method that is closely followed in this investigation.

The outcome shows that, if the second body could be treated as a particle of mass comparable to that of the first, the precessional motion of the equatorial plane of the spheroid would be synchronized with the motion of the node of the orbital plane, both moving in a retrograde direction. The corresponding period of revolution of the node would be short compared with that of the line of apsides. In the more general problem of two rigid spheroids, the motions of the orbital plane and of the equatorial planes are expressed by two fundamental periods, both of which are short in comparison with the period of apsidal revolution. The inclination of the orbital plane referred to the invariable plane of the system is found to be small compared with the inclinations of the equatorial planes of the spheroids.

In considering applications to actual binary systems, one has to keep in mind that real stars behave essentially as perfect fluids and therefore undergo mutual tidal distortion which will give rise to additional perturbations. These perturbations will largely determine the rate of apsidal advance. If, however, the tides can be regarded as an equilibrium phenomenon (that is, if the orbital period is long in comparison with periods of free nonradial oscillations of either component), the corresponding deformation will always be symmetrical with respect to the line joining the centers of both stars and the motions of the orbital plane and of the equatorial planes of both components should remain unaffected by tides to the first order in small quantities and be essentially the same as if the components were rigid.

Z. Kopal (Cambridge, Mass.).

Brouwer, Dirk, and Clemence, G. M. Numerical development of the disturbing function by correction of an approximate development. Astr. J. **52**, 64–67 (1946).

This paper describes a process by which a numerical development of the disturbing function (or its derivatives) of two celestial bodies moving in elliptic orbits may be conveniently obtained from any known development based on elements that differ but little from the adopted ones. Advantage is taken of the fact that the series development of the square of the distance D between two bodies is readily available from the Fourier series expressing the coordinates in elliptic motion in terms of the true, eccentric or mean anomaly. The series for D^2 are known to converge much more rapidly than those for the reciprocal of the distance; if the eccentric anomalies of both bodies are used as independent variables the expression for D is closed, containing only seven different arguments. The discussion is limited to a development of the negative odd powers of D, from which the disturbing function and its derivatives can be obtained by well-known processes. From the numerical point of view, the proposed method involves essentially series multiplications. The procedure has been adapted to punched card technique and is illustrated by an application to the perturbations of Mars by the earth. *Z. Kopal.*

Fessenkoff, B. G. On the possibility of capture at close passages of attracting bodies. Astr. J. Soviet Union [Astr. Zhurnal] **23**, 45–48 (1946). (Russian. English summary) [MF 16957]

The author considers the motion of an infinitesimal particle in the vicinity of a finite mass μ, which is small compared to the distant third body of the system, of mass $1-\mu$. He concludes that under the conditions postulated the infinitesimal particle cannot be captured by μ, since its orbit around μ is either a hyperbola or an ellipse whose semimajor axis is large compared to the size of the system of μ and $1-\mu$. *O. Struve* (Williams Bay, Wis.).

Krochmal, S. Sur une anomalie générale pour une orbite hyperbolique. Leningrad State Univ. Annals [Uchenye Zapiski] **82** [Math. Ser. **11**. Astronomy] [Publ. [Trudy] Observ. Astr. Univ. Leningrad **12**], 208–213 (1941). (Russian. French summary) [MF 16480]

Duarte, F. J. Analytic theory of the eclipses of the sun and of the occultation of stars by the moon (exposition of Bessel's method). Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. **8**, 907–936 (1944). (Spanish)

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. XI. Astrophys. J. **104**, 110–132 (1946).

[For part X see the same J. **103**, 351–370 (1946); these Rev. **7**, 494.] The paper continues part IX of this series [same J. **103**, 165–192 (1946); these Rev. **7**, 489]. A partially-polarized radiation field without axial symmetry is contemplated and the transfer equations established, allowing for Rayleigh scattering. The equations for the case of the diffuse reflection of a parallel beam of unpolarized radiation by a semi-infinite plane-parallel atmosphere are solved to the nth approximation. *G. C. McVittie* (London).

Ambarzumian, V. On the scattering and absorption in the atmospheres of planets. I. Leningrad State Univ. Annals [Uchenye Zapiski] **82** [Math. Ser. **11**. Astronomy] [Publ. [Trudy] Observ. Astr. Univ. Leningrad **12**], 64–85 (1941). (Russian. English summary) [MF 16479]

This paper deals with the "far interior" solutions of the equations of radiative transfer for plane parallel atmospheres. Thus, for the case

$$(*) \qquad \mu dI(\tau,\mu)/d\tau = I(\tau,\mu) - \tfrac{1}{2}\lambda \int_{-1}^{1} I(\tau,\mu')p(\mu',\mu)d\mu',$$

where λ is a positive constant ($\lambda \leqq 1$) and

$$p(\mu',\mu) = (2\pi)^{-1} \int_{-1}^{1} x(\mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}} \cos\varphi')d\varphi'$$

is the average over the azimuthal angle of the phase function $x(\cos\Theta)$ (which governs the probability that a pencil of radiation will be scattered in a direction inclined at an angle Θ to the direction of incidence), the author shows that

there exists a solution for the source function $B(\tau, \mu)$ of the form

$$(**) \qquad B(\tau, \mu) = \tfrac{1}{2}\lambda \int_{-1}^{1} I(\tau, \mu')p(\mu', \mu)d\mu' = b(\mu)e^{-k\tau},$$

where $b(\mu)$ and the characteristic root k are related through the functional equation

$$b(\mu) = \tfrac{1}{2}\lambda \int_{-1}^{1} p(\mu, \mu')b(\mu')(1+k\mu')^{-1}d\mu'.$$

When the equation of transfer (*) contains the additional term

$$-\tfrac{1}{4}\lambda F e^{-\tau/\mu_0} x(-\mu\mu_0 + (1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}}\cos\varphi)$$

on the right hand side, as would be the case when a parallel beam of radiation of net flux πF per unit area normal to itself is incident on the atmosphere in the direction $(-\mu_0, \varphi)$, the author further seeks a particular integral of the non-homogeneous equation of the form $Ce^{-\tau/\mu_0}$ which will be valid in the "mean." Finally, combining such particular integrals with solutions of the homogeneous equation of the form (**), an attempt is made to solve the problem of diffuse reflection approximately. [It may be noted that the method described in this paper has since been superseded by more powerful ones which provide exact solutions of the various problems: cf. Ambarzumian, Acad. Sci. USSR. J. Phys. **8**, 65–75 (1944); these Rev. **7**, 97; Chandrasekhar, Astrophys. J. **103**, 165–192 (1946); these Rev. **7**, 489; cf. also the preceding review.] *S. Chandrasekhar.*

Lebedinsky, A. Transfer of energy by convection currents in earth atmosphere and solar photosphere. Leningrad State Univ. Annals [Uchenye Zapiski] **82** [Math. Ser. **11**. Astronomy] [Publ. [Trudy] Observ. Astr. Univ. Leningrad **12**], 17–63 (1941). (Russian. English summary) [MF 16478]

Roy, Sunil Kumar. On polytropes rotating with variable angular velocity (in time). Proc. Nat. Acad. Sci. India. Sect. A. **12**, 1–8 (1942).

A star is supposed to describe small motions about a state of uniform rotation with angular velocity ω_0, the motion in meridian planes being wholly radial. The equation $p = K\rho^n$ is supposed to be satisfied throughout the star at all times. Motions such that the angular velocity is always uniform throughout the star are shown to be impossible. In oscillatory motion with period $\pi/\omega_0\lambda$, the angular velocity is shown to be a function of $r^2(\lambda^2 - \sin^2\theta)$, r and θ being polar coordinates with the axis of rotation as axis. *T. G. Cowling.*

Sen, H. K. Polytropic gas spheres with variable index. Proc. Nat. Acad. Sci. India. Sect. A. **12**, 13–36 (1942).

The polytropic index of a star is supposed to lie between n_1 and n_2, where $-1 < n_1 < n_2 < 5$. Certain properties of the star are shown to be intermediate between those of polytropes n_1, n_2, in particular, $\rho_c/\bar{\rho}$, the ratio β of gas to total pressure P, and $P/\rho^{1+1/m}$ $(0 < m \leqq 3)$. The results generalize those of Candler [Monthly Not. Roy. Astr. Soc. **100**, 14–24 (1939)] and Chandrasekhar [An Introduction to the Study of Stellar Structure, University of Chicago Press, 1939, chap. 3; Monthly Not. Roy. Astr. Soc. **99**, 673–685 (1939)]. *T. G. Cowling* (Bangor).

Sen, H. K. Radial oscillations of a variable star. Proc. Nat. Acad. Sci. India. Sect. A. **12**, 99–107 (1942).

The density outside a uniform core is supposed to vary like Kr^{-p}, where p is a positive integer. The failure of convergence of a series expansion at the surface is shown to imply that, if $p > 3$, no radial oscillations of the star are possible. [The argument appears to be unsound; if $p > 3$, Sen's equation (28) has a root greater than $1/\mu$, corresponding to (20)'s having a solution with radius of convergence less than $|x| = \mu$. Since (20) is applicable only when $x > \mu$, this solution cannot be disregarded.] *T. G. Cowling.*

Sen, H. K. Adiabatic pulsations of the Cepheid variable. Proc. Nat. Acad. Sci. India. Sect. A. **12**, 177–189 (1942).

Models are considered in which the density varies as a linear or quadratic function of the distance from the centre. Convergence arguments are again used to show that radial oscillations are impossible. The period of oscillation of a homogeneous star is shown to accord closely with observations of Cepheids. [The convergence argument is again unsound: certain solutions of Sen's equations converge beyond $|x| = 1$.] *T. G. Cowling* (Bangor).

Sen, H. K. Uniform radial oscillations of a star. Proc. Nat. Acad. Sci. India. Sect. A. **12**, 236–244 (1942).

A homogeneous star is shown to be the only one capable of radial oscillations such that the volume expansion is uniform throughout the star. This result, and the period of oscillation, are unaltered by rotation. In the discussion of rotating stars, the motion is supposed purely radial, Coriolis forces not being considered. *T. G. Cowling* (Bangor).

Sen, H. K. Radial oscillations of a slowly rotating star. Proc. Nat. Acad. Sci. India. Sect. A. **12**, 245–257 (1942).

Surfaces of equal density are assumed to be similar spheroids and to remain so during the oscillations. Purely radial motions are considered, Coriolis forces being omitted; other approximations are also made. The period of small oscillations is then found to be the same as for a nonrotating star, and large and purely radial oscillations are found to be impossible. *T. G. Cowling* (Bangor).

Sen, H. K. Radial oscillations of the generalised Roche's model. Proc. Nat. Acad. Sci. India. Sect. A. **13**, 44–53 (1943).

The model consists of a nucleus of uniform density with an atmosphere of infinitesimal mass. Small radial oscillations are shown to be impossible save for special ratios of the radius of the nucleus to that of the whole star; large oscillations are always impossible. [The discussion is invalid, since it does not consider the most general motion securing continuity of motion at the interface between the atmosphere and the core.] *T. G. Cowling* (Bangor).

Sen, H. K. The rotating Cepheid. Proc. Nat. Acad. Sci. India. Sect. A. **13**, 159–164 (1943).

A rapidly rotating uniform star is considered. Neglecting Coriolis forces, purely radial oscillations of uniform expansion and contraction, leaving the eccentricity of the surface spheroid unaltered, are shown to be impossible.
T. G. Cowling (Bangor).

Sen, H. K. Large radial oscillations of a star. Proc. Nat. Acad. Sci. India. Sect. A. **13**, 184–194 (1943).

The density of a star is supposed to be represented by a polynomial in the distance from the centre. A convergence argument is used to show that the model is incapable of large radial oscillations. [See the fifth preceding review.]
T. G. Cowling (Bangor).